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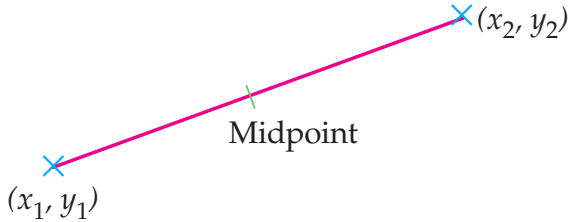
1.01 Midpoint of a line segment

A **line segment** is the part of a line that lies *between* two points. The **midpoint** of the line segment is the point that lies exactly half-way between the points.



In general, if the two points are (x_1, y_1) and (x_2, y_2) then the co-ordinates of the midpoint are:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Example

Find the midpoint of the line segment AB, where A = $(^{-}2, 3)$ and B = (5, 1).

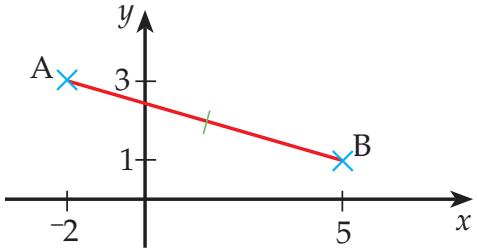
Answer

First draw a diagram to get a feeling for this question.

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{^{-}2 + 5}{2}, \frac{3 + 1}{2}\right)$$

$$= (1.5, 2)$$



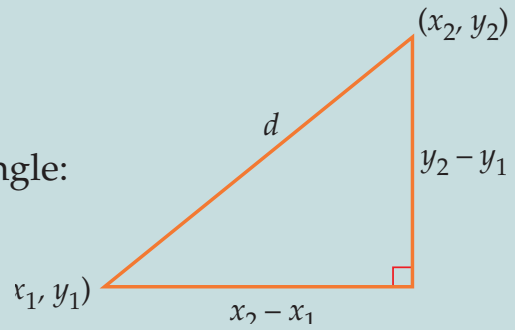
1.02 Distance between two points

Given two points on a graph, it is always possible to find the distance between them using Pythagoras.

In general, the given points are (x_1, y_1) and (x_2, y_2). These are joined up. The distance (d) between them can be calculated by finding the length of the hypotenuse of a right-angled triangle:

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

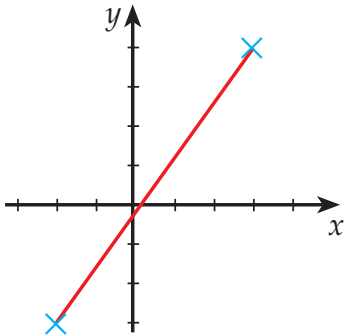


Example

Calculate the distance between the points $(^{-}2, ^{-}3)$ and $(3, 4)$.

Answer

$$
\begin{aligned}
d^2 &= (3 - {}^{-}2)^2 + (4 - {}^{-}3)^2 \\
&= 5^2 + 7^2 \\
&= 25 + 49 \\
&= 74 \\
d &= \sqrt{74} = 8.602 \ \ (4 \text{ sf})
\end{aligned}
$$

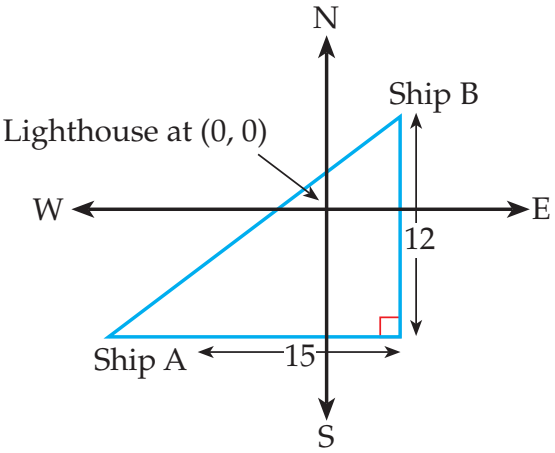


Example

Ship A is 8 km south and 10 km west of a lighthouse. Ship B is 4 km north and 5 km east of the same lighthouse. What is the shortest distance between the two ships?

Answer

Take the lighthouse as being at the origin (0, 0). Apply the directions N, E, S and W to the axes. Then ship A is at $(^{-}10, ^{-}8)$ and ship B is at (5, 4).



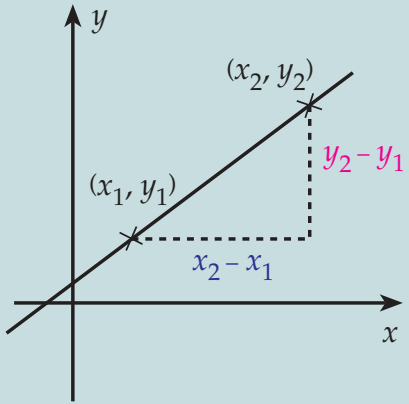
1.03 Gradients

The **gradient** of a line is a number which describes how steep the line is.

It is calculated from the fraction: $\frac{\text{change in } y}{\text{change in } x}$.

In general, the formula or rule giving the gradient of the line joining two points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

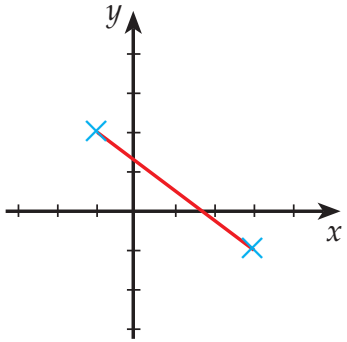


Example

Find the gradient of the line joining $(^{-}1, 2)$ and $(3, ^{-}1)$.

Answer

First draw a diagram to get a feeling for this question.



$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{^{-}1 - 2}{3 - ^{-}1} = \frac{^{-}3}{4}$$

We say that the gradient of a *vertical* line is **undefined**. The x-co-ordinates of any two points on a vertical line are the same. The gradient formula would involve dividing by 0, which is undefined.

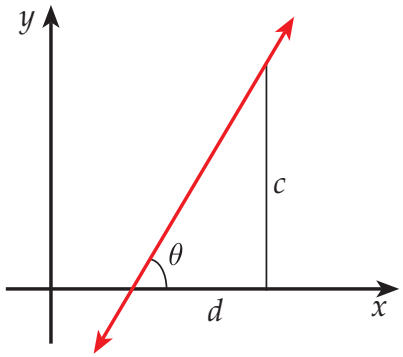
$$\text{Gradient} = \frac{r-q}{p-p} = \frac{r-q}{0}$$

1.04 Gradient of the line making an angle of θ with the x-axis

The diagram shows a line that makes an angle of θ with the positive direction of the x-axis.

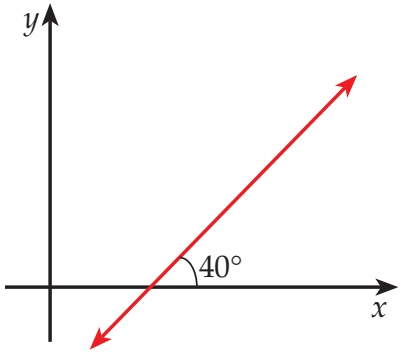
The gradient of this line is the fraction $\frac{c}{d}$. But from trigonometry, $\frac{c}{d} = \tan(\theta)$.

It follows that, if we know the angle that a line makes with the positive direction of the x-axis, the gradient of the line will be tan of this angle. And if we know the gradient of a line, we can use trigonometry ($\tan^{-1}$) to calculate the angle between the line and the x-axis.



Example

A line makes an angle of 40° with the x-axis, as shown in the diagram. Calculate the gradient of the line.

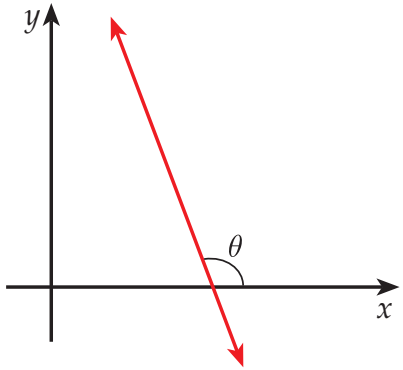


Answer

Gradient $= \tan(40°) = 0.84$.

Example

The line in the diagram has gradient $^{-}3$. Calculate the angle between the line and the positive direction of the x-axis. (Note: the gradient is negative, which means the angle is obtuse, i.e. greater than 90°.)



Answer

$$m = {}^{-}3$$
$$\tan(\theta) = {}^{-}3$$
$$\theta = {}^{-}71.6^\circ \text{ (using calculator)}$$

The required angle is 108.4°, obtained by adding 180° to the calculator value. In fact, the calculator gives the angle *below* the x-axis.

1.05 Rearranging line equations to the form $y = mx + c$

The previous information is only obvious if the straight-line equation is already in the form $y = mx + c$; i.e. if y is already the 'subject' of the equation. The following example shows how to rearrange a straight-line equation to obtain information about the gradient.

Example

Calculate the gradient of the line $5x - 3y + 2 = 0$. Hence find the angle the line makes with the positive direction of the x-axis.

Answer

Make y the subject, i.e. aim to get y by itself on the left-hand side of the equals sign.

$$5x - 3y + 2 = 0$$
$$-3y = -5x - 2$$
$$y = \frac{-5x}{-3} - \frac{2}{-3}$$
$$y = \frac{5x}{3} + \frac{2}{3}$$

This means the gradient is $\frac{5}{3}$.

The angle between this line and the positive direction of the x-axis is found by solving $\tan(\theta) = \frac{5}{3}$.

$$\tan(\theta) = \frac{5}{3} = 1.\dot{6}$$
$$\theta = 59.0° \quad (\tan^{-1} \text{ on a calculator})$$

1.06 Rearranging line equations to the form $ax + by + c = 0$

Although the form $y = mx + c$ is particularly convenient when working with gradients, not all straight lines can be expressed in this form; vertical lines cannot be. However, all straight lines *can* be written in the form $ax + by + c = 0$.

At this level of mathematics, there is an increasing emphasis on style and elegance. The convention we follow is that it is best to express straight-line equations in this form (i.e. $ax + by + c = 0$), with a, b and c as integers, so that the equation has no fractions. The x-term should be written first, and with a positive coefficient (number multiplying x).

Example

Express $y = 2x - 5$ in the form $ax + by + c = 0$.

Answer

$$y = 2x - 5$$
$$-2x + y + 5 = 0$$
$$2x - y - 5 = 0$$

Example

Express $y = \frac{-2x}{3} + 4$ in the form $ax + by + c = 0$.

Answer

$$y = \frac{-2x}{3} + 4$$
$$3y = -2x + 12$$
$$2x + 3y - 12 = 0$$

1.07 Equation of a line – point and gradient

Given the gradient m and a point (x_1, y_1) on a line, the equation of the line is:

$y - y_1 = m(x - x_1)$.

This is often called the **point–gradient equation**.

Example

Find the equation of the line that passes through $(4, {}^{-}3)$ with gradient 2.

Answer

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - {}^{-}3 &= 2(x - 4) \\ y + 3 &= 2x - 8 \\ y &= 2x - 11 \\ \text{or} \quad 2x - y - 11 &= 0 \end{aligned}$$

1.08 Equation of a line – two points

Given two points (x_1, y_1) and (x_2, y_2) on a line, the gradient, m, is:

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

This is often called the **point–point equation**.

Calculate the gradient first, and then use the formula in **1** above. This gives:

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1).$$

An alternative expression for the point–point equation is:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example

Find the equation of the line joining the points $(^{-}2, 1)$ and $(3, 2)$.

Answer

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 1}{3 - {}^{-}2} = \frac{1}{5}$$

Now substitute into $y - y_1 = m(x - x_1)$:

$$y - 1 = \frac{1}{5}(x - {}^{-}2)$$

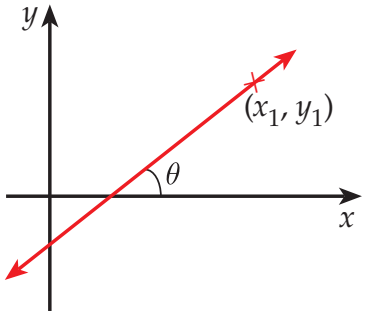
$$5y - 5 = x + 2 \qquad \text{(cross-multiplying and simplifying)}$$

$$5y - x - 7 = 0$$

$$x - 5y + 7 = 0$$

1.09 Equation of a line – point and angle

Given the angle θ with the x-axis and a point on the line, if the angle that the line makes with the positive direction of the x-axis is θ, then the **gradient** of the line must be **tan(θ)**. Use this information to substitute into the equation $y - y_1 = m(x - x_1)$.



Example

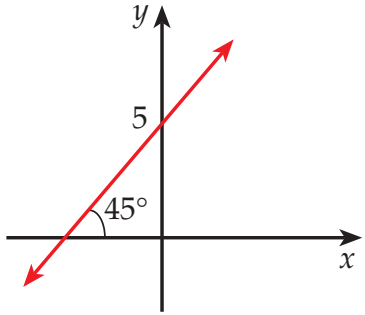
Find the equation of the line that has a y-intercept of 5 and crosses the x-axis at an angle of 45°.

Answer

The gradient of the line $= \tan(45°) = 1$. The line passes through (0, 5). Substitute into $y - y_1 = m(x - x_1)$:

$$y - 5 = 1(x - 0)$$
$$y - 5 = x$$
$$y = x + 5$$

or $x - y + 5 = 0$



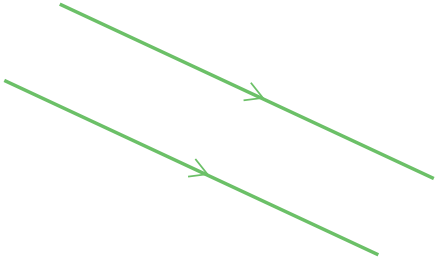
2.01 Parallel lines

Parallel lines have the same direction. This means they never meet.
Parallel lines (in two dimensions) have the *same gradient*.

Consider the two lines $y = m_1x + c_1$ and

$$y = m_2x + c_2.$$

If $m_1 = m_2$ then the two lines are parallel.



Example

Are the two lines (1) $y = \frac{3x}{4} - 5$ and

(2) $4y - 3x + 10 = 0$ parallel?

Answer

By inspection, the gradient of line (1) is $\frac{3}{4}$.

Rearranging the equation for line (2) to the form $y = mx + c$, we obtain $4y = 3x - 10$, which is equivalent to the equation $y = \frac{3}{4}x - \frac{5}{2}$. This means line (2) also has gradient $\frac{3}{4}$.

So, the two lines are parallel.

Example

What is the equation of the line that passes through the point $(6, {}^{-}2)$ and is parallel to the line $2x - 3y + 4 = 0$?

Answer

Rearrange the equation of the given line:

$$2x - 3y + 4 = 0$$
$${}^{-}3y = {}^{-}2x - 4$$
$$y = \frac{{}^{-}2x}{{}^{-}3} - \frac{4}{{}^{-}3}$$
$$y = \frac{2x}{3} + \frac{4}{3}$$

The gradient of this line is $\frac{2}{3}$. Therefore, the required line has gradient $\frac{2}{3}$ and passes through $(6, {}^{-}2)$.

The equation of the line is found by substituting into $y - y_1 = m(x - x_1)$:

$$y - {}^{-}2 = \frac{2}{3}(x - 6)$$
$$3(y + 2) = 2(x - 6)$$
$$3y + 6 = 2x - 12$$
$${}^{-}2x + 3y + 18 = 0$$
$$2x - 3y - 18 = 0$$

2.02 & 2.03 Perpendicular lines

Two lines are **perpendicular** if they meet at right-angles. The gradients multiply to $^{-}1$. Each gradient is the negative reciprocal of the other.

If the two lines are perpendicular:

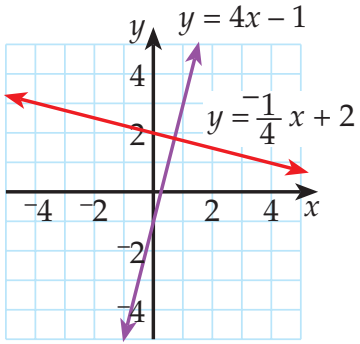
$$m_1 \times m_2 = {}^{-}1.$$

In particular, this means $m_1 = \dfrac{{}^{-}1}{m_2}$ and $m_2 = \dfrac{{}^{-}1}{m_1}$.

There is one exception to this rule. A horizontal line (gradient 0) and a vertical line (gradient undefined) are perpendicular to each other. In this case, $m_1 \times m_2$ has no value.

Example

Verify that the two lines in the diagram are perpendicular.



Answer

The two gradients are

$m_1 = 4$ and $m_2 = \frac{-1}{4}$.

$m_1 \times m_2 = 4 \times \frac{-1}{4} = -1$.

Example

Find the gradient of a line perpendicular to the line $x + 3y - 4 = 0$.

Answer

The given line can be rearranged to $3y = {}^{-}x + 4$, or

$y = \frac{{}^{-}1}{3}x + \frac{4}{3}$, which has gradient $\frac{{}^{-}1}{3}$.

A line perpendicular to this one would have gradient 3.

Note: $\frac{{}^{-}1}{3} \times 3 = {}^{-}1$ as required.

The gradient properties for perpendicular lines can be used to determine the equations of lines that are perpendicular to a given line and pass through a given point.

Example

Find the equation of the line that passes through (4, 6) and is perpendicular to the line passing through the points $(2, {}^{-}3)$ and $({}^{-}1, 5)$.

Answer

The gradient of the line through $(2, {}^{-}3)$ and $({}^{-}1, 5)$ is given by the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

That is, $m = \dfrac{5 - {}^{-}3}{{}^{-}1 - 2} = \dfrac{8}{{}^{-}3} = \dfrac{{}^{-}8}{3}$.

The gradient of a perpendicular line would be $\frac{3}{8}$.

Note: $\frac{3}{8}$ is the negative reciprocal of $\frac{-8}{3}$.

The required line has gradient $\frac{3}{8}$ and passes through (4, 6). The equation is:

$$y - 6 = \frac{3}{8}(x - 4)$$

$$8y - 48 = 3x - 12$$

$$3x - 8y + 36 = 0$$

Example

Give the equation of the line perpendicular to $y = 2x + 6$ that has the same x-intercept.

Answer

Two pieces of information are required:

1 the gradient

2 the point the line passes through.

Gradient $= \frac{-1}{2}$ (product of gradients is -1).

The x-intercept for the line is when $y = 0$:

$0 = 2x + 6 \qquad x = -3$

i.e. x-intercept is $(-3, 0)$.

Substitute into $y - y_1 = m(x - x_1)$ to obtain the equation:

$$y - 0 = \frac{-1}{2}(x - -3)$$
$$2y = -1(x + 3)$$
$$2y = -x - 3$$
$$x + 2y + 3 = 0$$

2.04 Collinear points

Points are **collinear** if they all lie on the same line. This means that there should be no 'bend' as the points are joined and, in particular, the gradient between any two pairs of points should be the same.

Example

Show that the points A $(^{-}3, ^{-}11)$, B $(0, ^{-}2)$ and C $(2, 4)$ are collinear.

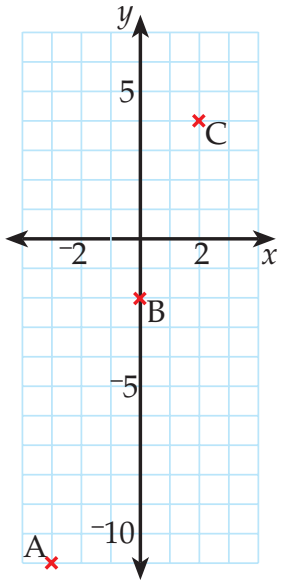
Answer

Answer this question by showing that gradient AB = gradient BC.

$$\text{Gradient AB} = \frac{^{-}2 - ^{-}11}{0 - ^{-}3} = \frac{9}{3} = 3$$

$$\text{Gradient BC} = \frac{4 - ^{-}2}{2 - 0} = \frac{6}{2} = 3$$

From A to B to C, there is no change in the gradient, and so the three points A, B and C are collinear.



2.05 & 2.06 Drawing lines – the gradient-intercept method

If the equation is given in the form:

$y = mx + c$

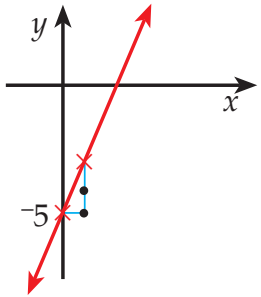
then m is the gradient, and c is the y-intercept (the place where the line cuts the y-axis).

To draw a line using this method, we locate the y-intercept. Make sure that the equation is in the form $y = mx + c$ first. Then, from the y-intercept, draw a line that has a gradient of m. In general, this means 'go along one to the right, and then up or down by m' (up if m is positive, down if m is negative).

If the gradient is a fraction, it is sometimes easier to draw a line with this gradient by going along to the right by the denominator (lower number) of the fraction, and then up or down by the numerator (upper number).

Example

Draw the line $y = 2x - 5$.



Answer

The equation is already in the form $y = mx + c$.

$m = 2$, $c = {}^{-}5$

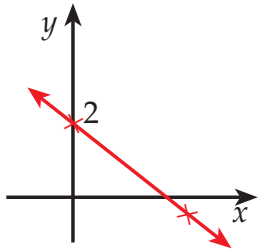
The y-intercept is $^{-}5$, i.e. $(0, {}^{-}5)$. Mark this point with a cross.

From the cross, draw a line with a gradient of 2 (i.e. along 1, up 2).

Mark this point with another cross. Join the two points up, extending the line in both directions.

Example

Draw the line $3x + 4y - 8 = 0$.



Answer

First rearrange to the form $y = mx + c$.

$3x + 4y - 8 = 0$

$$4y = {}^{-}3x + 8$$

$$y = \frac{{}^{-}3x}{4} + \frac{8}{4}$$

$$y = \frac{{}^{-}3x}{4} + 2$$

$m = \frac{{}^{-}3}{4}, c = 2$

The y-intercept is 2, i.e. (0, 2). Mark this point with a cross.

From the cross, draw a line with a gradient of $\frac{{}^{-}3}{4}$ (i.e. along 4, down 3). Mark this point with another cross.

Join the two points up, extending the line in both directions.

Two special cases are **vertical lines** and **horizontal lines**.

1 Vertical lines have equations of the form $x = c$. The line passes through the point $(c, 0)$.

2 Horizontal lines have equations of the form $y = c$. The line passes through the point $(0, c)$.

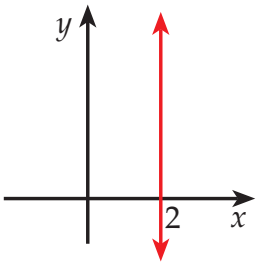
Example

Draw the following lines.

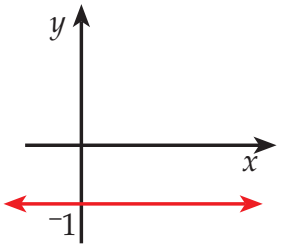
a $x = 2$ **b** $y = {}^{-}1$

Answer

a



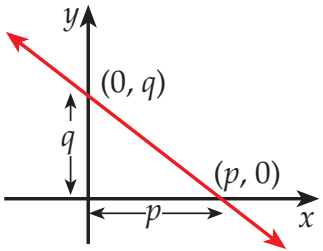
b



2.07 Drawing lines – the intercept–intercept method

When the line equation is given in the form:

$ax + by + c = 0$



it is sometimes easier to find the x- and y- intercepts and then join them to form the line, than it is to rearrange and work out the gradient and y-intercept.

- The ***x*-intercept** is the point where the line cuts the x-axis. The co-ordinates of any point on the x-axis are $(p, 0)$. In other words, at the x-intercept, the y-value $= 0$.
- Similarly, the ***y*-intercept** is of the form $(0, q)$, i.e. at the y-intercept, the x-value is 0.

This suggests the following approach. Given a line equation in the form $ax + by + c = 0$:

1 to work out the x-intercept, substitute $y = 0$, and solve the equation for x

2 to work out the y-intercept, substitute $x = 0$, and solve the equation for y.

Example

Find the x- and y-intercepts for the line $3x - 2y + 12 = 0$, and hence draw the graph.

Answer

For the x-intercept, substitute $y = 0$:

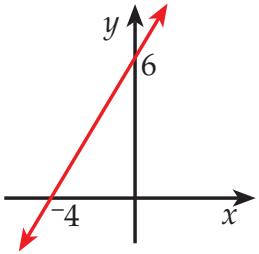
$$3x - 2 \times 0 + 12 = 0$$
$$3x = {}^{-}12$$
$$x = {}^{-}4$$

i.e. $({}^{-}4, 0)$ is the x-intercept.

For the y-intercept, substitute $x = 0$:

$$3 \times 0 - 2y + 12 = 0$$
$${}^{-}2y = {}^{-}12$$
$$y = 6$$

i.e. $(0, 6)$ is the y-intercept.



Plot these two intercepts, join them up and extend the line in both directions.

2.08 Writing the equation of a line given the graph

If you are given the graph of a line, there should always be enough information on the diagram to identify *two* points on the line.

Once the two points are known, it is best to use the **point–point equation**: $y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$. Another way of expressing this equation is:

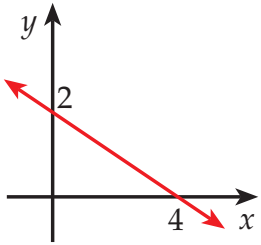
$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

How is this second equation formed?

We are making two equivalent gradient statements. They must be equal to each other. Creating a single statement from these gradient expressions gives us the equation.

Example

Write the equation of the line shown in the diagram.



Answer

Clearly, two points on the line are the x-intercept $(4, 0)$ and the y-intercept $(0, 2)$.

The equation is:

$$\frac{y-0}{x-4} = \frac{2-0}{0-4}$$

$$\frac{y}{x-4} = \frac{2}{-4}$$

$$-4y = 2(x-4) \quad \text{(cross-multiplying)}$$

$$-4y - 2x + 8 = 0$$

$$2x + 4y - 8 = 0$$

$$x + 2y - 4 = 0$$

2.09 Intersection of lines

Some topics in mathematics blend algebra and geometry together. When working out where lines intersect, you can use:

- geometry – draw the lines and inspect the graph to locate the point of intersection, or
- algebra – solve a pair of simultaneous equations, where each equation represents one of the lines.

Ideally, when locating the intersection of two lines, consider a combined approach:

- drawing a graph to get an approximate idea of the location
- using algebra to determine the exact co-ordinates.

Pairs of lines do not always intersect on the grid (with integer values) and, in some cases, only algebra will give you the exact values.

Example

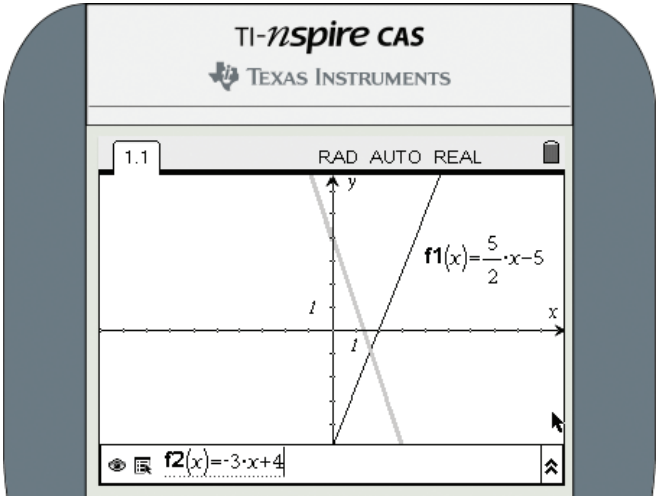
Determine the point of intersection of the two lines:

$5x - 2y - 10 = 0$

$3x + y - 4 = 0.$

Answer

Use technology or otherwise to draw the graph of both lines on the same grid.



The graph shows that the lines intersect near the point $(1.5, -1)$. We can use the 'Trace' function on a CAS calculator to obtain a very accurate estimate.

Note that the technology may require you to enter the equations of the lines in the form of $y = mx + c$:

$y = \frac{5}{2}x + 5$ (1)

$y = {}^{-}3x + 4$ (2)

We then use algebra to obtain this result exactly. Solve the simultaneous equations:

$5x - 2y = 10$ (1)

$3x + y = 4$ (2)

Multiply equation (2) by 2:

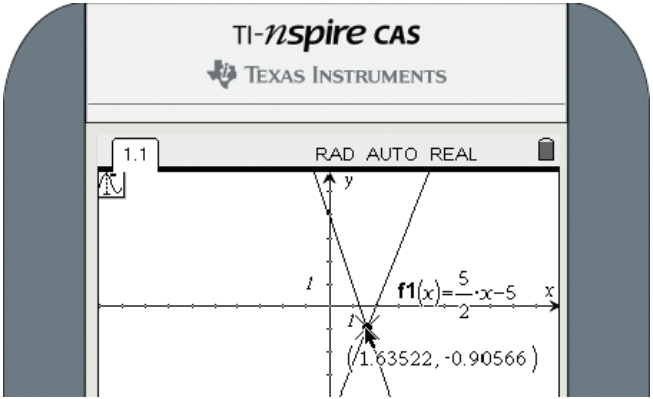
$5x - 2y = 10$ (1)

$6x + 2y = 8$ (2)

Add the equations:

$11x = 18$

$$x = \frac{18}{11} = 1\frac{7}{11}$$



Substitute back into equation (2) to obtain the value of y:

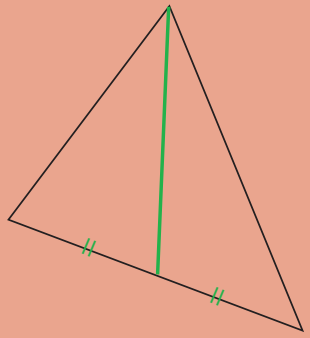
$$\begin{aligned} y &= {}^{-}3x + 4 \\ &= {}^{-}3x \times \frac{18}{11} + 4 \qquad (2) \\ &= \frac{{}^{-}54}{11} + \frac{44}{11} \\ &= \frac{{}^{-}10}{11} \end{aligned}$$

The co-ordinates of the point of intersection are $\left(1\frac{7}{11}, \frac{{}^{-}10}{11}\right)$.

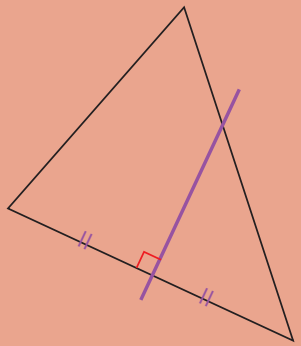
2.10 Triangle properties and proofs

Co-ordinate geometry can be used to obtain the equations of three special types of line in triangles.

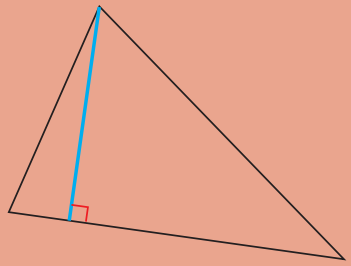
A **median** is the line that joins a **vertex** of a triangle to the **midpoint** of the opposite side.



A **perpendicular bisector** is the line that passes through the **midpoint** of a side and is **perpendicular** (at right-angles) to that side.



An **altitude** is the line that joins a **vertex** of a triangle to the opposite side, and is **perpendicular** to that side.



This table summarises the features of each type of line. Each line has two of the three properties.

Line	Passes through a **vertex**	Passes through the **midpoint** of a side	Is **perpendicular** to a side
Median	✓	✓	×
Perpendicular bisector	×	✓	✓
Altitude	✓	×	✓

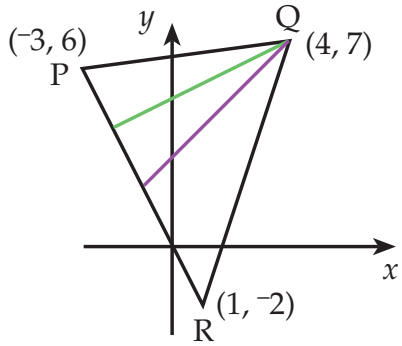
Example

Triangle PQR is drawn here.

Find the equation of:

a the **median** from PR to Q

b the **altitude** to Q from side PR.



Answer

a The required **median** passes through Q and the midpoint of PR. The midpoint is given by:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-3 + 1}{2}, \frac{6 + -2}{2}\right) = (-1, 2).$$

The median is the line that passes through $(-1, 2)$ and $(4, 7)$.

The gradient is $\frac{7 - 2}{4 - -1} = \frac{5}{5} = 1$.

The equation is $y - 7 = 1(x - 4)$

$$y = x - 4 + 7$$

$$y = x + 3$$

b The gradient of PR is $\frac{6 - {}^{-}2}{{}^{-}3 - 1} = \frac{8}{{}^{-}4} = {}^{-}2$. The **altitude** is perpendicular to PR, so has a gradient of $\frac{1}{2}$.

The altitude is a line that passes through (4, 7) with gradient $\frac{1}{2}$.
The equation is:

$$y - 7 = \frac{1}{2}(x - 4)$$

$$y = \frac{1}{2}x - 2 + 7$$

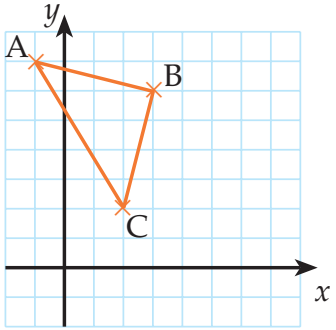
$$y = \frac{1}{2}x + 5$$

It is also possible to use co-ordinate geometry to prove that triangles are isosceles or right-angled.

Example

Show that the triangle with vertices A = $(^{-}1, 7)$, B = (3, 6) and C = (2, 2) is right-angled and isosceles.

Answer



Draw the triangle to help answer the question.

Gradient of AB $= \frac{6-7}{3-{}^{-}1} = \frac{{}^{-}1}{4}$.

Gradient of BC $= \frac{2-6}{2-3} = \frac{{}^{-}4}{{}^{-}1} = 4$.

The product of these two gradients is $^{-}1$, so the lines AB and BC are perpendicular, and therefore $\angle ABC = 90°$.

Use the distance formula to calculate the length of AB and the length of BC.

For AB:

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - {}^{-}1)^2 + (6 - 7)^2} \\ &= \sqrt{4^2 + ({}^{-}1)^2} \\ &= \sqrt{17} \end{aligned}$$

For BC:

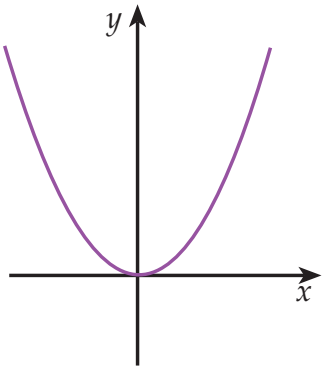
$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2 - 3)^2 + (2 - 6)^2} \\ &= \sqrt{({}^{-}1)^2 + ({}^{-}4)^2} \\ &= \sqrt{17} \end{aligned}$$

Length of AB = length of BC, so the triangle is isosceles (has two sides the same length).

3.01 Properties of parabolas

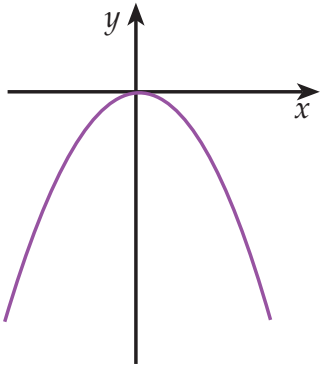
1 $y = x^2$

The basic parabola is the one for $y = x^2$ drawn here. Any other parabola will relate to this one in some way.



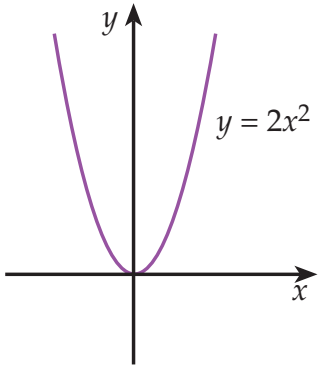
2 $y = {}^{-}x^2$

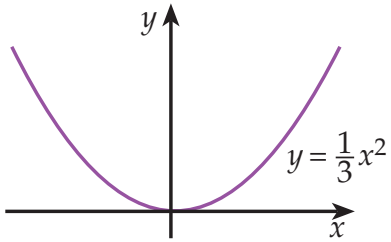
If there is a minus sign in front of the x^2 term, the parabola will be inverted (upside-down). The basic example is the graph of $y = {}^{-}x^2$ (drawn here).



3 $y = ax^2$

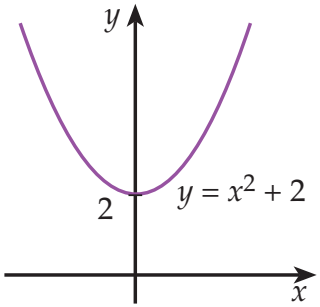
Numbers in front of the x^2 alter the steepness of the parabola. Two examples are $y = 2x^2$ and $y = \frac{x^2}{3}$ or $\frac{1}{3}x^2$.

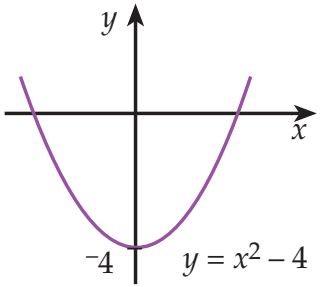


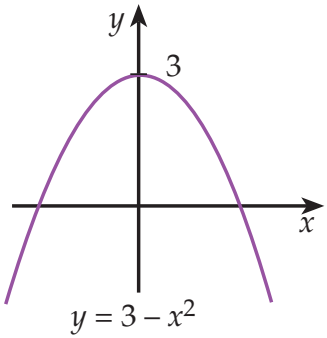


4 $y = x^2 \pm c$ $(c > 0)$

Adding or subtracting a number from x^2 moves the basic parabola up or down.

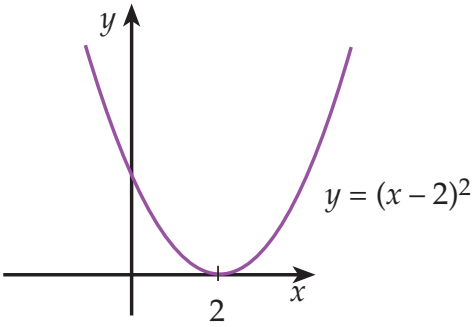


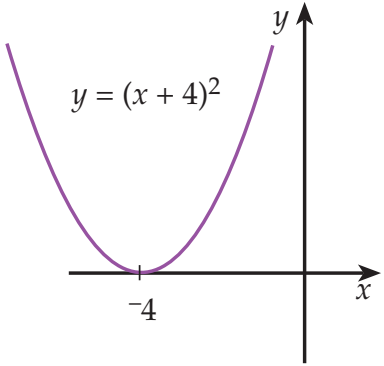


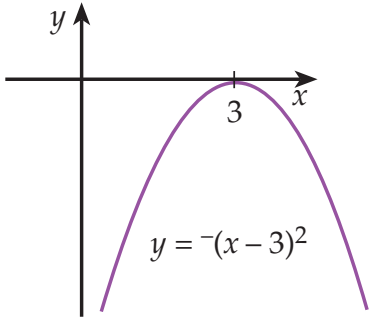


5 $y = (x \pm c)^2$ $(c > 0)$

Adding or subtracting a number to or from x before squaring (i.e. in brackets) moves the basic parabola sideways. Adding moves the basic parabola to the left, and subtracting moves it to the right. Always check by substituting the x-value of the vertex (bottom point). You should get 0.







6 $y = (x \pm c)^2 \pm d$

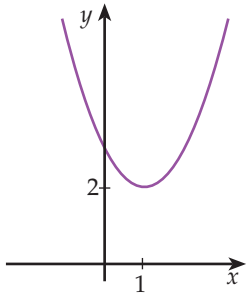
The types mentioned in **4** and **5** above can occur together.

Example

Draw the graph of $y = (x - 1)^2 + 2$.

Answer

This will be a basic parabola shifted 1 to the right, and then moved upward 2 units. The vertex of the parabola is at the point (1, 2).



3.02 Parabolas in factorised form

When a quadratic function is supplied already factorised, it is particularly easy to work out the intercepts. The parabola will have one or two x-intercepts, and one y-intercept.

- The x-intercepts are the one or two numbers that make the expressions in the brackets equal to 0.
- The y-intercept is the constant term when the brackets are expanded (multiplied out). Alternatively, substitute $x = 0$ and work out y.
- The vertex (bottom point) of the parabola will have its x-co-ordinate exactly half-way between the x-intercepts. To calculate the y-co-ordinate of the vertex, substitute the x-value into the quadratic.

Collectively, the intercepts and the vertex of the parabola are known as **key features** of the parabola.

Example

Draw the graph of $y = (x + 1)(x - 3)$.

Answer

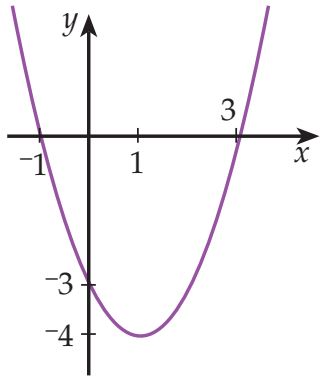
x-intercepts are at $^{-}1$ and 3.

Expand $(x + 1)(x - 3)$ to get $x^2 - 2x - 3$.

The y-intercept is at $^{-}3$.

The vertex is half-way between $^{-}1$ and 3, i.e. at $x = 1$.

The y-co-ordinate of the vertex is at $y = (1 + 1)(1 - 3) = 2 \times {}^{-}2 = {}^{-}4$.



Example

Draw the graph of $y = x(2 - x)$.

Answer

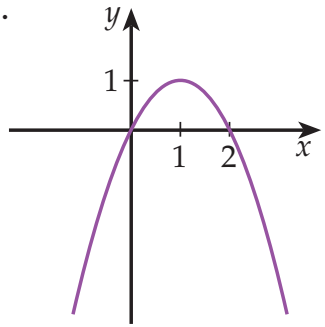
x-intercepts are at 0 and 2.

The y-intercept is found by putting $x = 0$ to get $0(2 - 0) = 0$.

The vertex is half-way between 0 and 2, i.e. at $x = 1$.

The y-co-ordinate of the vertex is $y = 1(2 - 1) = 1$.

Note: the parabola is upside-down. This is to be expected; when expanded, the quadratic function starts with $^{-}x^2$.

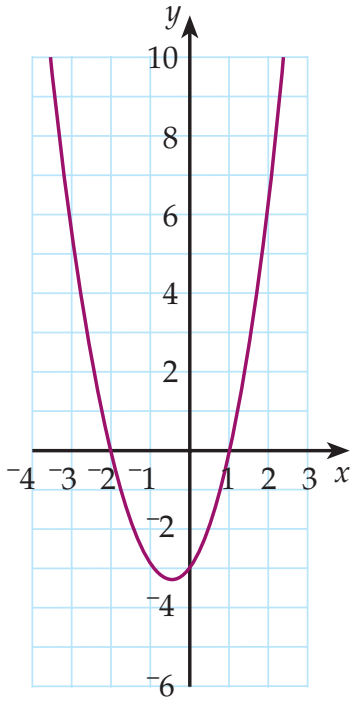


Always label the key features on a parabola.

Some parabolas in factorised form may not be similar to the basic parabola given by $y = x^2$. In these cases, information from a point on the parabola that is not an x-intercept will be given.

Example

Write the equation for the parabola drawn here.



Answer

The parabola has x-intercepts at $^{-}2$ and 1, so its equation will be of the form $y = a(x + 2)(x - 1)$.

The point (2, 6) lies on the graph so $x = 2$ and $y = 6$ must satisfy its equation. This is the information we need to work out a.

Substituting into $y = a(x + 2)(x - 1)$:

$$6 = a \times (2 + 2) \times (2 - 1)$$

$$6 = 4a$$

$$a = \frac{6}{4} = \frac{3}{2} = 1.5$$

The equation of the parabola is

$$y = \frac{3}{2}(x + 2)(x - 1) \text{ or } y = 1.5(x + 2)(x - 1).$$

3.04 Cubics in factorised form

A **cubic** is a polynomial with x^3 as its highest power of x. In graph questions, a cubic will usually be given to you in factorised form – in general as:

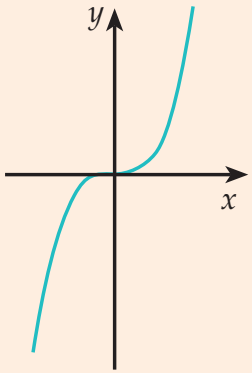
$y = (x - a)(x - b)(x - c)$.

A cubic has three factors. This means that if the factors are all different, the graph of the cubic will cross the x-axis in three places.

KEY POINTS

The basic cubic $y = x^3$

The graph of $y = x^3$ should be learned, but is not particularly helpful for drawing other cubics unless all three factors are the same.

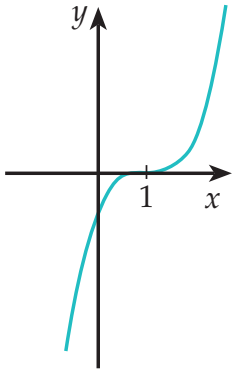


Example

Draw the graph of $y = (x - 1)^3$.

Answer

The graph is the same as the basic cubic but moved 1 unit to the right.

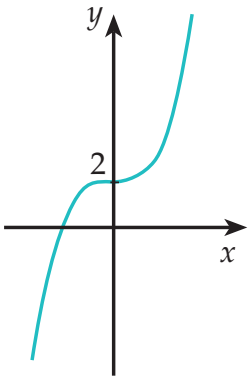


Example

Draw the graph of $y = x^3 + 2$.

Answer

The graph is the same as the basic cubic but moved 2 units upwards.



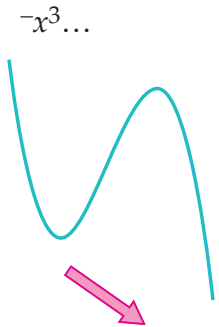
Note that:

- if the equation starts with x^3 when expanded, the general trend is for the graph to go up from left to right,
- if the equation starts with $^{-}x^3$, the graph will trend down from left to right.

The factors of the cubic give us the x-intercepts. The x-intercepts are the three numbers that result when the expressions inside each bracket are equated to 0.

The y-intercept is the number that results when 0 is substituted into the cubic instead of x.

Upward trend
Up–Down–Up



Downward trend
Down–Up–Down

Example

Draw the graph of the cubic $y = (x + 1)(x - 1)(x - 2)$.

Answer

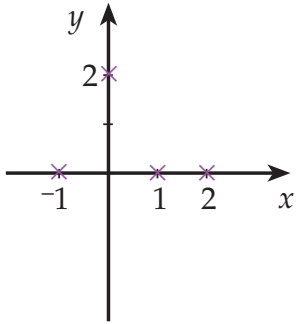
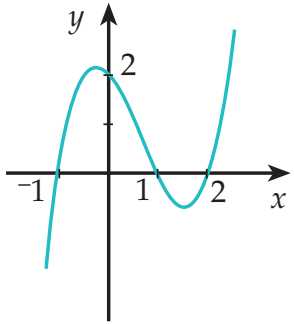
x-intercepts are $^{-}1$, 1 and 2.

y-intercept is:

$$y = (0 + 1)(0 - 1)(0 - 2)$$
$$= 1 \times {}^{-}1 \times {}^{-}2$$
$$= 2$$

Plot these points, and impose the up-down-up pattern on them to obtain a smooth curve.

Up–Down–Up

Pattern +  Points =  Graph

Two special cases

1 If one factor is x, the cubic passes through the origin $(0, 0)$.

Example

Draw the graph of $y = x(2x + 1)(1 - x)$.

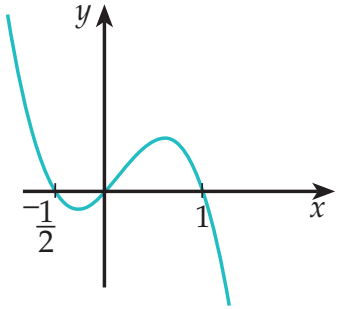
Answer

x-intercepts are given by:

$x = 0$ $\quad 2x + 1 = 0$ $\quad 1 - x = 0$

$2x = {}^{-}1$ $\quad {}^{-}x = {}^{-}1$

$x = \frac{{}^{-}1}{2}$ $\quad x = 1$



The three x-intercepts are $\left\{\frac{{}^{-}1}{2}, 0, 1\right\}$.

The y-intercept is

$0 \times (2[0] + 1) \times (1 - 0) = 0 \times 1 \times 1 = 0$.

Note: if this cubic were expanded, it would start with ${}^{-}2x^3$.

So, the general trend is *downwards* from left to right.

2 If one factor is repeated, the graph of the cubic *touches* the x-axis instead of cleanly cutting through. The repeated factor may be written twice, or may be a squared bracket.

Example

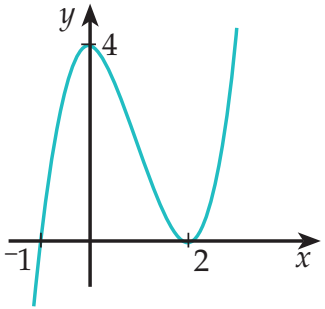
Draw the graph of $y = (x - 2)^2(x + 1)$.

Answer

x-intercepts are at 2 (twice) and $^-1$.

y-intercept is at $(0 - 2)^2(0 + 1) = 4 \times 1 = 4$.

If expanded, the cubic would start with x^3. So, the general trend is *upwards* from left to right.



You don't have to draw these graphs extremely accurately or exactly to scale. What is important is to represent the shape and to label the key features, such as the intercepts, correctly.

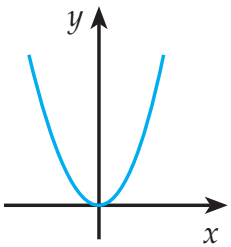
4.01 Testing for functions

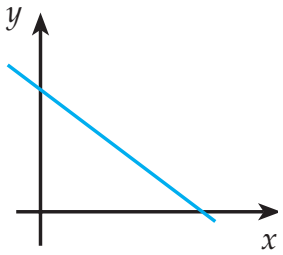
A function has the property that each x-value has only one y-value. We can look at a mathematical graph and decide whether it represents a function. To do this, we use the **vertical line test**.

A graph represents a function if any vertical line cuts the graph at only one point.

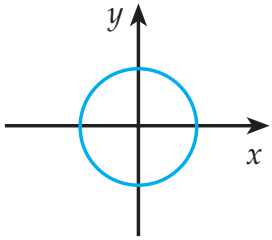
Example

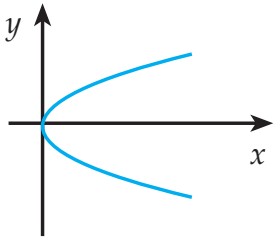
1 Functions





2 Not functions





If a vertical line crosses the graph anywhere more than once, then the graph does not represent a function.

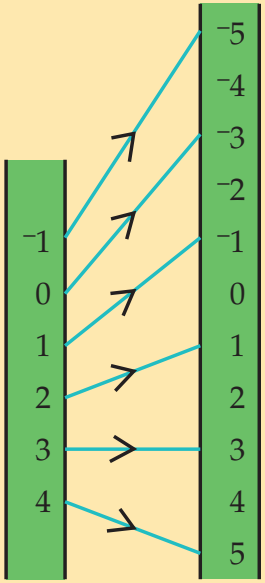
One to one, many to one

The idea of getting a unique result is central to the idea of functions. Sometimes, relations can be expressed as sets of ordered pairs (these give information about where you would plot the graph on a set of x- and y-axes) or as an arrow diagram.

For example, some values that satisfy the relation $f(x) = 2x - 3$ (or $y = 2x - 3$) can be expressed as either a set:

$\{\ldots, (^{-}1, ^{-}5), (0, ^{-}3), (1, ^{-}1), (2, 1), (3, 3), (4, 5), \ldots\}$

or as an arrow diagram (on the right).

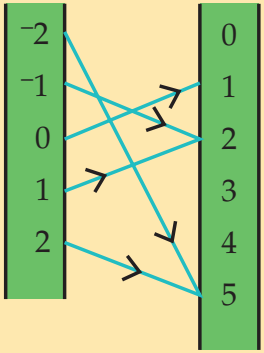


This function is an example of a **one-to-one** relation. Each x-value has a unique y-value associated with it, and each y-value has only one x-value that produces it.

The other kind of function is **many to one**. Only one arrow leaves any particular x-value, but several x-values can give the same y-value.

Here's an example of a many-to-one function:

$\{\ldots, (^{-}2, 5), (^{-}1, 2), (0, 1), (1, 2), (2, 5), \ldots\}$.
(This is a listing of several pairs from the function $f(x) = x^2 + 1$.)



The arrow diagram on the right shows 'many' (in this case, two) x-values leading to one y-value.

Other relations can be classified as 'one to many' or 'many to many' and these are not functions.

In this terminology, 'many' means 'several' or 'more than one', rather than 'a lot'.

4.02 Function notation

A very convenient way to represent the function process is to use a new notation, i.e. to write a function as $f(x)$.

The x in brackets represents the number that is put into the function machine. Because the number put in can take many values, we often call it a *variable*. We can then write the formula for the function as:

$$f(x) = \ldots$$

Example

The function machine shown in the first example on page 50 was $4x + 5$. Write this function in $f(x)$ notation.

Answer

$f(x) = 4x + 5$

f(*x*) notation is very useful in a branch of mathematics called calculus.

Using $f(x)$ notation is particularly convenient when we want to represent the process of putting a certain number into a function. For example, $f(4)$ means that 4 is substituted for x, and the result calculated. It is important to realise that when a number is substituted into $f(x)$, only one answer can result – the answer is unique.

Examples

$f(x) = 3x - 5$

a Calculate $f(^{-}4)$.

b Solve the equation $f(x) = 13$.

Answers

a
$$\begin{aligned} f(x) &= 3x - 5 \\ f(^{-}4) &= 3 \times {}^{-}4 - 5 \\ &= {}^{-}12 - 5 \\ &= {}^{-}17 \end{aligned}$$

b
$$\begin{aligned} f(x) &= 13 \\ 3x - 5 &= 13 \\ 3x &= 18 \\ x &= 6 \end{aligned}$$

4.03 Domain and range

In many cases, x (the number put into the function) can be any real number – a whole number, an integer, a fraction, a decimal, negative, positive – in fact, any number from the real numbers.

- The domain of a function is the set of numbers that can be put into the function. These are usually x-values.
- The range of a function is the set of all possible numbers that result. These are y or $f(x)$ values.

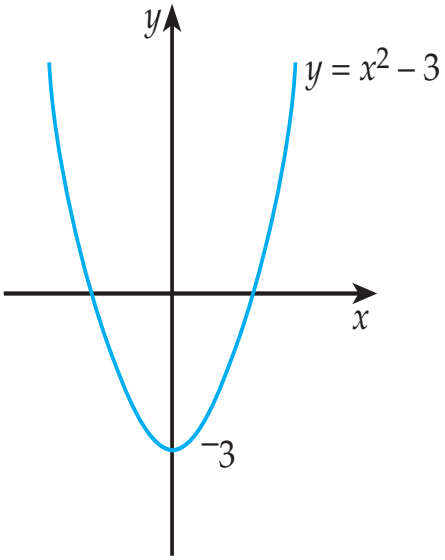
However, sometimes there are restrictions on the values that x can take.

Example

State the domain and range of the function $y = x^2 - 3$.

Answer

Draw the graph:



- The domain is the set of all real numbers. We write this as $\mathbb{R}$. There are no restrictions on what x-values can be substituted into $x^2 - 3$. Every value on the x-axis corresponds to a point on the graph of the function.
- The range is $y \geq -3$. The graph only has y-values greater than or equal to -3. Squaring a number and subtracting 3 never gives an answer less than -3.

Sometimes, the nature of the function itself restricts the domain, or x-values, to only some real numbers. For example, we know that it is not possible to calculate the square root of a negative number, or to divide by 0.

Examples

Write the domain of these functions:

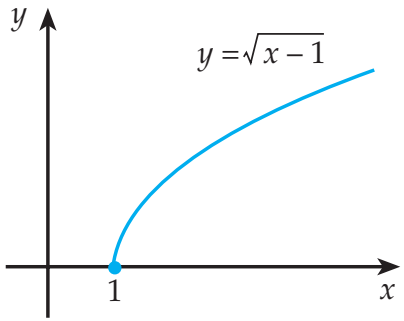
a $f(x) = \sqrt{x-1}$

b $y = \dfrac{1}{x+4}$.

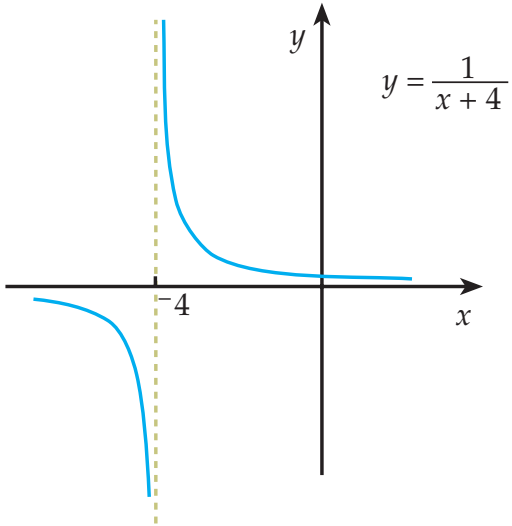
Answers

a The domain of $f(x) = \sqrt{x-1}$ is $x \geq 1$. The function is undefined if x is a real number less than 1.

The graph of $y = \sqrt{x-1}$ is only defined for x-values that are greater than or equal to 1.



b The domain of $y = \frac{1}{x + 4}$ is $x \neq -4$; that is, all values of x except for -4. The function has no value when $x = -4$ because division by 0 is undefined.



Here is a method that helps determine the domain and range from a graph.

Slide a ruler up the *x*-axis (for the domain) or along the *y*-axis (for the range). Look for where the ruler crosses the graph. These places give *x*- and *y*-values that are in the domain and range respectively.

4.04 Absolute value

The absolute value of a number can be thought of as its size or the distance it is from 0 on a number line. We always take distance as being *non-negative*.

The **absolute value** function is written as $f(x) = |x|$.

- If $x < 0$, then $|x| = {}^{-}x$, making it positive.
- If $x \geq 0$, then $|x| = x$ (so that it stays non-negative).

The effect of the absolute value brackets $|\ \ldots\ |$ is that they make the result of the expression inside positive if it is negative.

Examples

Evaluate the following.

a $|{}^{-}3|$ **b** $|14|$ **c** $|10 - 17|$

Answers

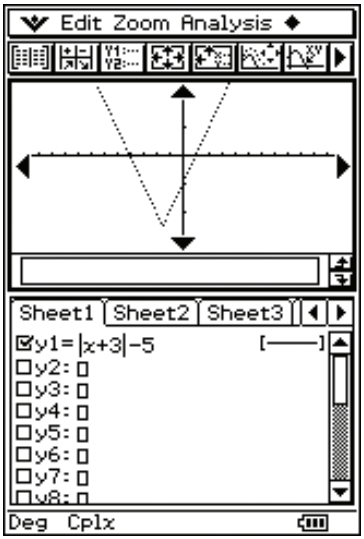
a 3 **b** 14 **c** 7

Example

Use graphing software to draw the graph of $f(x) = |x + 3| - 5$. Identify the key features of the graph, including the domain, the range and any intercepts.

Answer

On a CAS calculator:



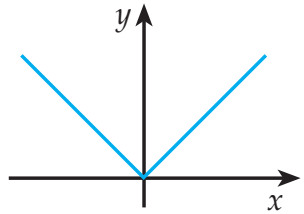
Note that the command for absolute value is ABS( ).

The term 'graphing software' here is very broad. It could include the use of graph-drawing programs, a graphics calculator, a spreadsheet or a CAS calculator. The graphs can also be drawn by hand.

The graph of the absolute value function has only non-negative y-values:

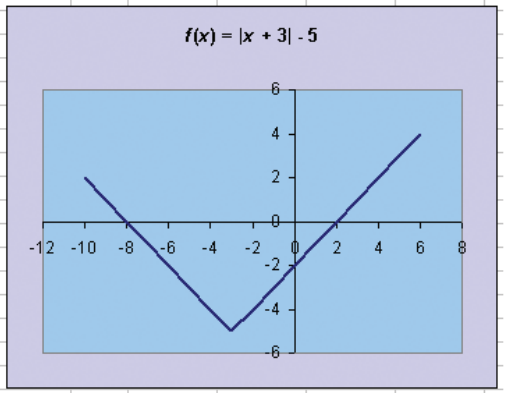
Note that the domain, or set of possible x-values, for $|x|$ is the real numbers, $\mathbb{R}$.

The range of $f(x) = |x|$ is $\{y: y \geq 0, y \in \mathbb{R}\}$.



From a spreadsheet:

	A	B	C	D	E	F	G	H
1	Table of values and graph for $f(x) = \|x + 3\| - 5$							
2								
3	x	$\|x + 3\| - 5$						
4	-10	2						
5	-9	1						
6	-8	0						
7	-7	-1						
8	-6	-2						
9	-5	-3						
10	-4	-4						
11	-3	-5						
12	-2	-4						
13	-1	-3						
14	0	-2						
15	1	-1						
16	2	0						
17	3	1						
18	4	2						
19	5	3						
20	6	4						



Note that the formula for the absolute value of the number in any cell, e.g. B1, is =ABS(B1).

The domain is all the real numbers, $\mathbb{R}$, because there are no restrictions on the values of x.

The range is $y \geq {}^{-}5$.

The y-intercept is $(0, {}^{-}2)$.

The x-intercepts are $({}^{-}8, 0)$ and $(2, 0)$.

The vertex of this absolute value graph is $({}^{-}3, {}^{-}5)$.

4.05 Square root

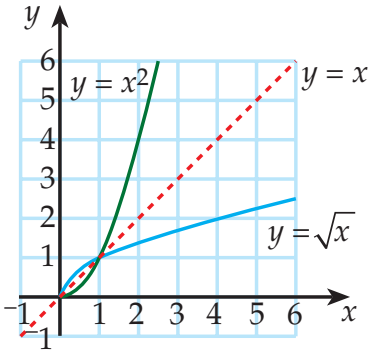
The square root function $f(x) = \sqrt{x}$ is the inverse of the function $y = x^2$.

Here we take the square root as meaning the positive square root only. Remember that we are dealing with functions here so, for any x-value that we can substitute, we want a unique y-value as the result.

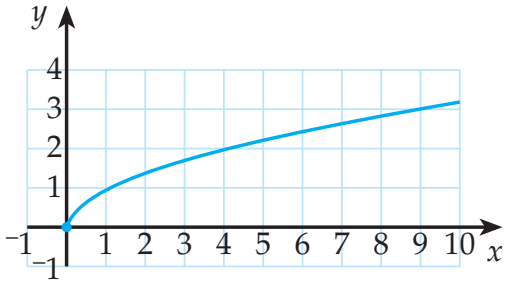
We know that taking the square root is the operation that undoes squaring. For example, $7^2 = 49$, so $\sqrt{49} = 7$. This means that, if (a, b) is a point on the graph of $y = x^2$, then (b, a) must lie on the graph of $y = \sqrt{x}$.

So, to obtain the graph of the square root function, all we need to do is to swap the x- and y-values of the co-ordinates. On the graph, we do this by reflecting the graph of the parabola $y = x^2$ in the line $y = x$.

This diagram shows the result of reflecting $y = x^2$ in the line $y = x$:



The graph of $y = \sqrt{x}$ by itself is shown below. As you would expect, it passes through points such as (0, 0), (1, 1), (4, 2), (9, 3) and so on.



KEY POINTS

The graph of $f(x) = \sqrt{x}$ starts at the origin (0, 0). The filled circle at (0, 0) shows that this point is included (see the diagram at bottom right on the previous page).

The **domain** of the function $f(x) = \sqrt{x}$ is all positive real numbers, as well as 0; that is, the non-negative real numbers. We write the domain as $x \geq 0$ or, more formally, as $\{x : x \geq 0, x \in \mathbb{R}\}$.

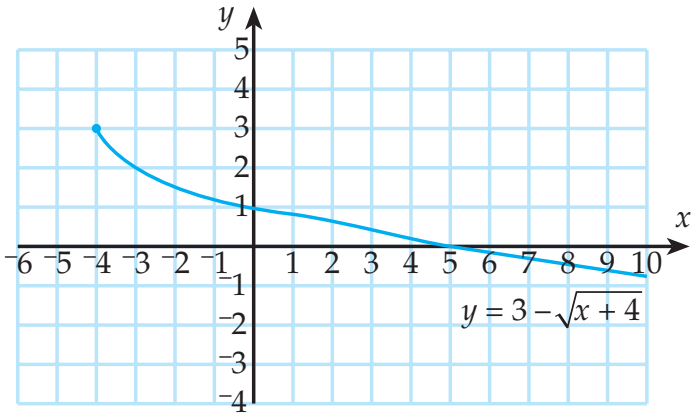
The **range** of the function $f(x) = \sqrt{x}$ is all positive real numbers, as well as 0; that is, the same as the domain. We write the range as $y \geq 0$ or, more formally, as $\{y : y \geq 0, y \in \mathbb{R}\}$.

In the real number system, we cannot evaluate square roots of negative numbers, so we have to restrict the domain to $x \geq 0$. Likewise, we are dealing with a function here, so the answer is the positive square root, which means the range is $y \geq 0$.

Example

Use graphing software to draw the graph of $f(x) = 3 - \sqrt{x + 4}$.
Identify the key features of the graph, including the domain, the range and any intercepts.

Answer



Note that the key features can be identified by inspecting the graph, but can also be explained mathematically.

Any x-value greater than or equal to $^{-}4$ can be substituted into $f(x)$ because that means $x + 4$ is non-negative, so a square root can be calculated. The **domain** is $x \geq {}^{-}4$.

The **range** is $y \leq 3$. When subtracting a non-negative number from 3, the largest value you can ever get is 3 itself, but any value less than 3 is always possible.

The ***y*-intercept** is found by setting $x = 0$:

$$\begin{aligned} y &= f(0) \\ &= 3 - \sqrt{0 + 4} \\ &= 3 - \sqrt{4} \\ &= 3 - 2 \\ &= 1 \end{aligned}$$

The ***x*-intercept** is found by setting $y = f(x) = 0$ and solving the resulting equation:

$$
\begin{aligned}
f(x) &= 0 \\
3 - \sqrt{x+4} &= 0 \\
\sqrt{x+4} &= 3 \\
x + 4 &= 9 \qquad \text{(squaring both sides)} \\
x &= 5
\end{aligned}
$$

5.01 Simple hyperbolas

Any mathematical relation where $x \times y =$ number has a graph called a **hyperbola**.

Examples

$xy = 6$ $xy = 1$ $xy = {}^{-}12$ etc.

All the points where the x-value and y-value multiply to the given number k lie on the hyperbola graph.

In general, the equation of a hyperbola is:

$$xy = k.$$

Example

Draw the graph of $xy = 6$ by plotting points and joining them with a smooth curve.

Answer

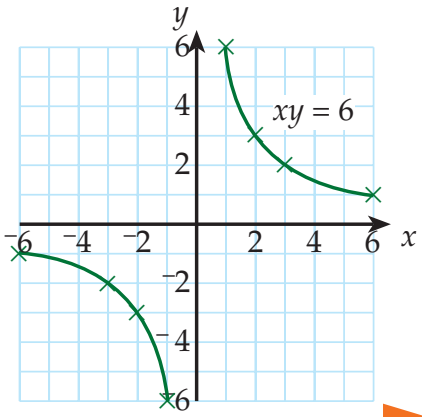
First, write all the pairs of numbers that multiply to give 6. Some number pairs will be negative.

Note: there are other pairs of numbers, which are not integers, e.g. 5 and 1.2, or 0.5 and 12, etc.

Graphing these points, we get *two distinct branches*.

x	y
1	6
2	3
3	2
6	1

x	y
-1	-6
-2	-3
-3	-2
-6	-1



Hyperbolas *always* have two branches. You must draw both branches unless the context of the question means some values can be excluded.

Key characteristics of hyperbolas

1. All hyperbolas have two distinct branches. The graph of a hyperbola is not continuous.
2. The branches of a hyperbola get very close to the x- and y-axes, without ever actually touching these lines. A line that a graph gets very close to for extreme values is called an **asymptote**. Note: in this section, the x- and y-axes happen to be asymptotes, but it is also possible for other lines to be asymptotes. Asymptotes are not part of the graph itself, but are drawn in as an indication only.
3. An alternative way of writing the equation for a hyperbola is to make y the subject.

   $xy = k$ is equivalent to $y = \frac{k}{x}$.

Example

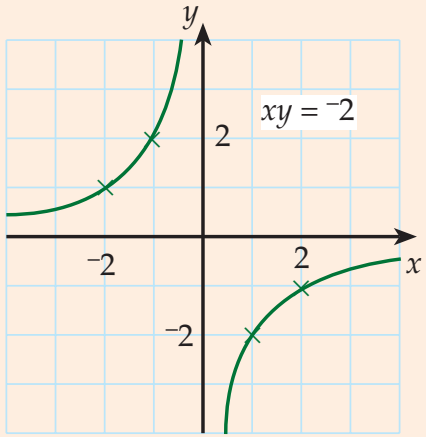
$xy = 4$ is equivalent to $y = \frac{4}{x}$.

4 When x and y multiply to a *negative* number, the hyperbola still has two branches but they are in the other two quarters of the diagram.

Example

Draw the graph of $xy = {}^{-}2$.

Answer



5.02 Translations of the hyperbola

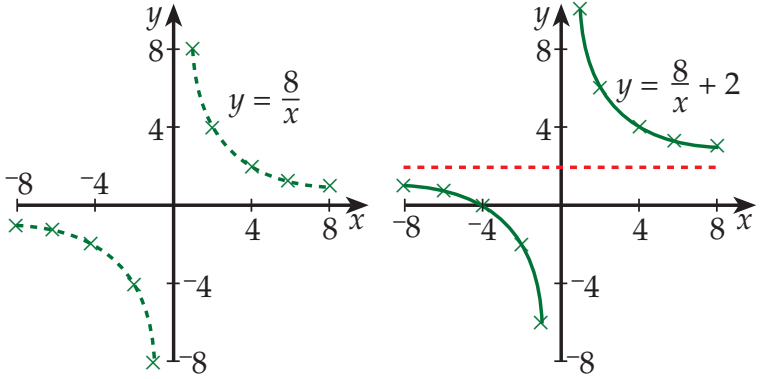
Vertical translations

Consider the hyperbola given by $y = \frac{8}{x} + 2$.

After dividing 8 by x, the result has 2 added to it.

This moves each y-value up by 2 compared with the graph of $y = \frac{8}{x}$.

Important note: the horizontal asymptote has also moved up by 2, i.e. the equation of the horizontal asymptote is $y = 2$.



In general, the graph of $y = \frac{k}{x} + a$ is the same shape as the graph of $y = \frac{k}{x}$ but moved up by a units.

Example

Draw the graph of $y = \frac{-4}{x} - 3$. Write the co-ordinates of the x-intercept.

Answer

This graph has the same shape as the graph of $y = \frac{-4}{x}$ but moved down 3 units.

To calculate the x-intercept, substitute $y = 0$.

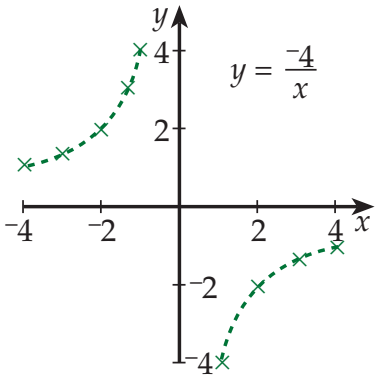
$$0 = \frac{-4}{x} - 3$$

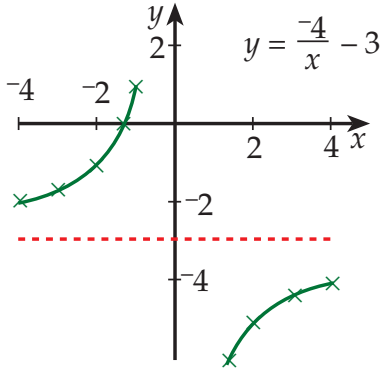
$$3 = \frac{-4}{x}$$

$$3x = -4$$

$$x = \frac{-4}{3} \quad \text{or} \quad -1\frac{1}{3}$$

The x-intercept is $\left(-1\frac{1}{3}, 0\right)$.





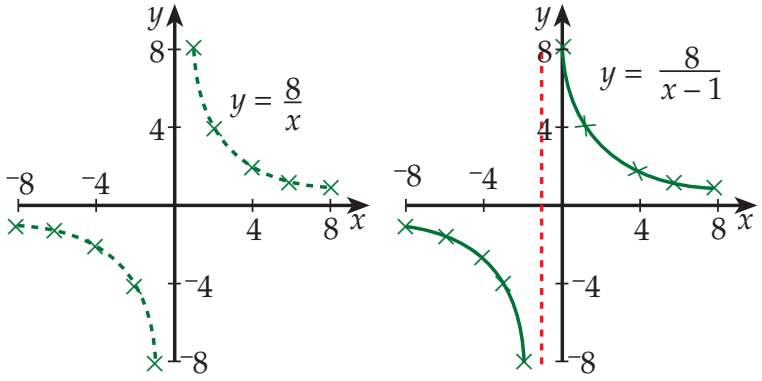
Horizontal translations

Consider the hyperbola given by $y = \frac{8}{x+1}$.

The effect of adding on 1 *before* dividing into 8 is not immediately obvious, so we will investigate the properties of this graph by plotting points.

x	$y = \frac{8}{x+1}$
-5	-2
-3	-4
-2	-8
-1	undefined
0	8
1	4
3	2
7	1

Here is the graph, with the graph of $y = \frac{8}{x}$ as a comparison.



It should be clear that the graph of $y = \frac{8}{x}$ has been moved *1 unit to the left*. Note that $y = \frac{8}{x+1}$ is undefined when $x = -1$. This is what we would expect, because there is an asymptote at $x = -1$.

This property, of having an asymptote for the value of x that makes the denominator 0, is very useful when drawing the graphs of hyperbolas.

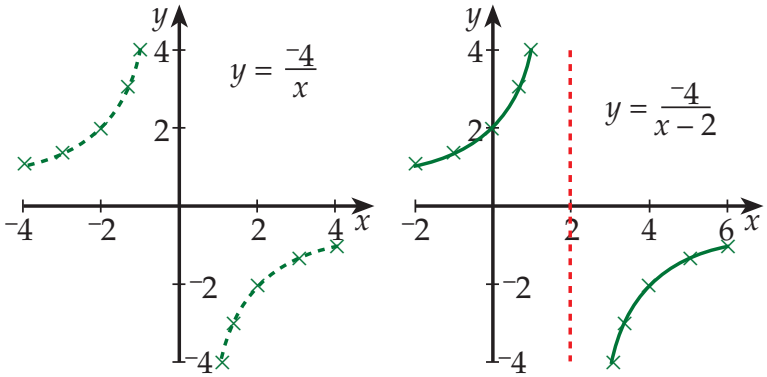
Example

Draw the graph of $y = \frac{-4}{x-2}$.

State the equations of both asymptotes and write the co-ordinates of the y-intercept.

Answer

This graph is the same as the graph of $y = \frac{-4}{x}$, but moved 2 units to the right.



The asymptotes to the graph of $y = \frac{-4}{x-2}$ are $y = 0$ (the horizontal asymptote) and $x = 2$ (the vertical asymptote).

To calculate the y-intercept, substitute $x = 0$.

$$y = \frac{-4}{x-2}$$
$$= \frac{-4}{0-2} = \frac{-4}{-2}$$
$$= 2$$

The y-intercept is $(0, 2)$.

In general, the graph of $y = \frac{k}{x \pm b}$ $(b > 0)$ is the same shape as the graph of $y = \frac{k}{x}$ but:

- moved to the left by b units if the denominator is $x + b$
- moved to the right by b units if the denominator is $x - b$.

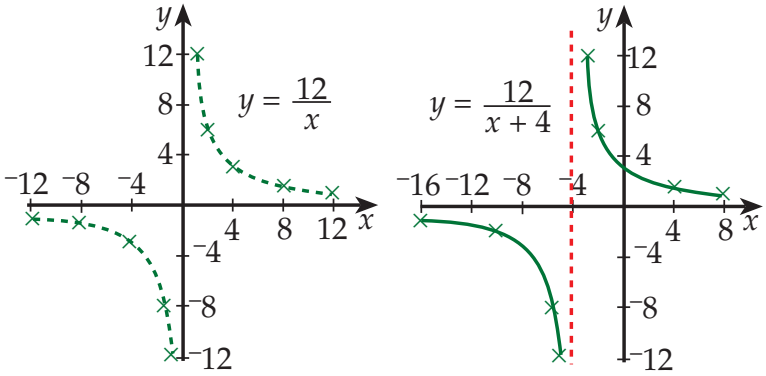
Example

Draw the graph of $y = \frac{12}{x+4}$.

State the equations of both asymptotes and give the co-ordinates of any intercepts.

Answer

This graph is the same as the graph of $y = \frac{12}{x}$, but moved 4 units to the left.



The asymptotes to the graph of $y = \frac{12}{x+4}$ are $y = 0$ (the horizontal asymptote) and $x = -4$ (the vertical asymptote).

There is no x-intercept (because the x-axis is an asymptote) and the y-intercept is (0, 3).

Both horizontal and vertical translations

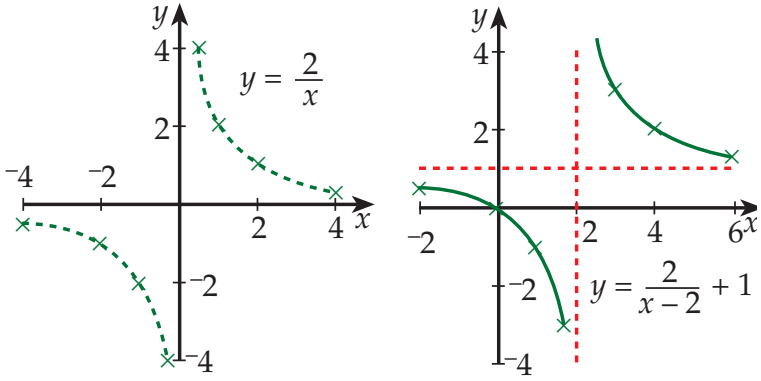
It is possible to have both horizontal and vertical translations operating in the same question. Sometimes, there will need to be some rearrangement of the given equation before this is obvious.

Example

Draw the graph of $y = \frac{2}{x-2} + 1$.

Answer

The basic shape of this hyperbola is that of $y = \frac{2}{x}$. It has been moved 1 unit upwards and 2 units to the right. The graph of $y = \frac{2}{x}$ has been drawn as a comparison.



Both asymptotes have moved also: the new asymptotes are $y = 1$ (the horizontal asymptote) and $x = 2$ (the vertical asymptote).

Example

Draw the graph of the hyperbola given by $y = \frac{6}{x-1} - 2$.
Give the equations of both asymptotes and write the co-ordinates of any intercepts.

Answer

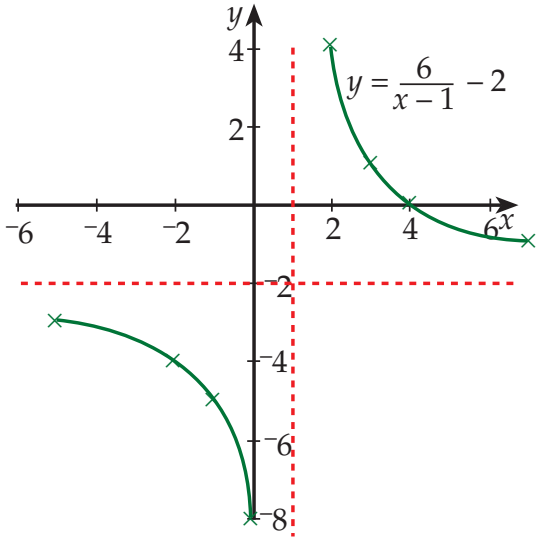
The basic shape is that of $y = \frac{6}{x}$ but moved 1 unit to the right and 2 units down.
The two asymptotes are $x = 1$ and $y = {}^{-}2$.

For the x-intercept, substitute $y = 0$:

$$0 = \frac{6}{x-1} - 2$$
$$2 = \frac{6}{x-1}$$
$$2x - 2 = 6$$
$$2x = 8$$
$$x = 4$$

For the y-intercept, substitute $x = 0$:

$$y = \frac{6}{0-1} - 2$$
$$y = {}^{-}6 - 2$$
$$y = {}^{-}8$$



5.03 Finding the equation of the hyperbola given the graph

If sufficient information is given with the graph of a hyperbola, it is possible to write the equation of the hyperbola. At this level, the general equation of a hyperbola will be expressed in the form:

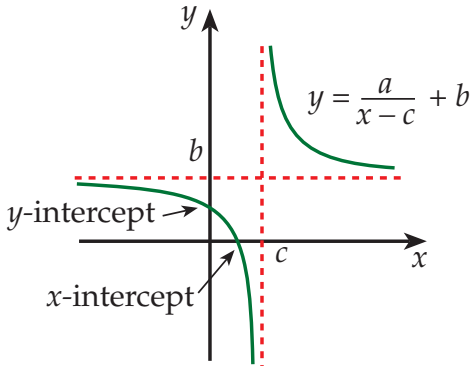
$$y = \frac{a}{x - c} + b.$$

Useful information includes:

- the equation of the **vertical asymptote** (in the form $x = c$)
- the equation of the **horizontal asymptote** (in the form $y = b$)
- any **intercepts** on the axes
- the co-ordinates of any **points on the hyperbola**.

b and c can be either positive or negative numbers, so it is not necessary to write the general form as $y = \frac{a}{x \pm c} \pm b$.

The connection between the information on the graph and the equation of the hyperbola (in the form $y = \frac{a}{x - c} + b$) is as follows:

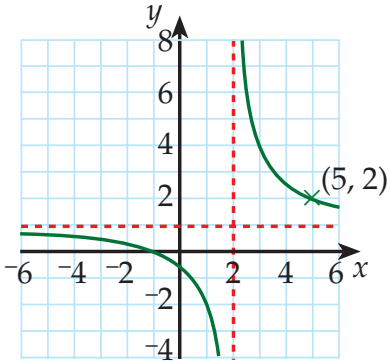


- c is the value given in the equation of the vertical asymptote, that is, $x = c$.
- b is the value given in the equation of the horizontal asymptote, that is, $y = b$.
- a is calculated by substituting b and c and the co-ordinates of any intercept, or point on the hyperbola, into the equation $y = \frac{a}{x - c} + b$.

- Why does $x = c$ give the vertical asymptote? The hyperbola is undefined for one x-value only, the value that gives the vertical asymptote. This occurs when dividing by zero in the fraction, that is when $x - c = 0$, so when $x = c$.
- Why does $y = b$ give the horizontal asymptote? The horizontal asymptote is the line $y = b$, because when larger and larger x-values are substituted into $\frac{a}{x - c}$, this fraction approaches 0, and we are left with y being very close to b.

Example

Write the equation of this hyperbola.



Answer

The general form is $y = \frac{a}{x - c} + b$.

The vertical asymptote is $x = 2$, so $c = 2$.

The horizontal asymptote is $y = 1$, so $b = 1$.

Substitute $b = 1$ and $c = 2$ into the equation:

$$y = \frac{a}{x - 2} + 1.$$

Now note that the graph passes through (5, 2), so $x = 5$ and $y = 2$ satisfy the equation:

$$2 = \frac{a}{5 - 2} + 1$$

$$2 = \frac{a}{3} + 1$$

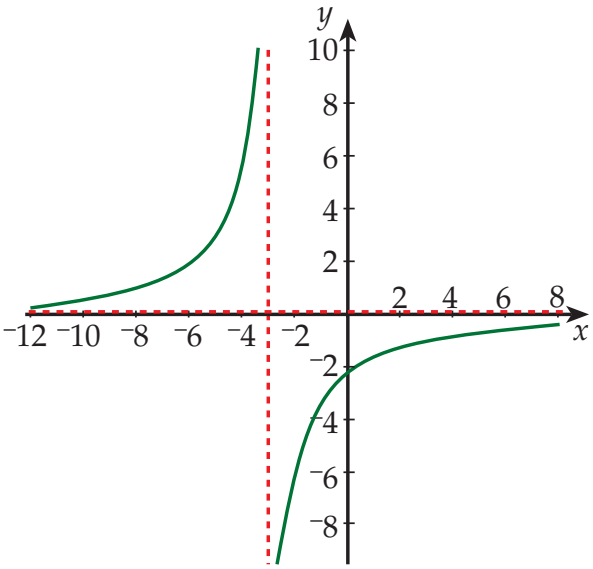
$$\frac{a}{3} = 1$$

$$a = 3$$

The equation of the hyperbola is $y = \frac{3}{x - 2} + 1$.

Example

Write the equation of this hyperbola.



Answer

The general form is $y = \frac{a}{x - c} + b$.

The vertical asymptote is $x = -3$, so $c = -3$, and the denominator in the hyperbola equation will be $x + 3$.

The horizontal asymptote is $y = 0$, so $b = 0$.

Substitute $b = 0$ and $c = -3$ into the equation: $y = \frac{a}{x + 3}$.

Now note that the y-intercept is $^{-}2$, so $x = 0$ and $y = {}^{-}2$ satisfy the equation:

$$^{-}2 = \frac{a}{0 + 3}$$

$$^{-}2 = \frac{a}{3}$$

$$a = {}^{-}6$$

The equation of the hyperbola is $y = \dfrac{^{-}6}{x + 3}$.

5.05 Exponential graphs

We use powers to represent repeated multiplication.

In the example, 5 is called the **base**, and 3 is the **index**, or **exponent**.

In general, the form is $(\text{base})^{\text{index}}$.

Example

$5^3 = 5 \times 5 \times 5 = 125$

The number a is called the **base**, and x is called the **exponent**.
Note: $a > 0$ and $a \neq 1$.

Any function of the form $y = a^x$ is said to be an **exponential** function.
There is a number in the base, and x in the exponent, or index.

Example

$y = 2^x$

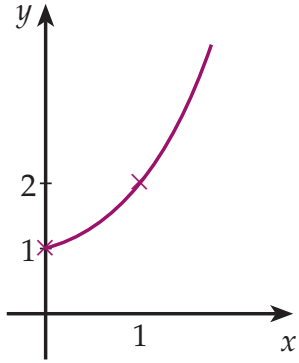
2^x is not the same as x^2. For example $2^3 = 8$, but $3^2 = 9$.

Growth curves

Exponential functions have graphs called growth curves. This is because when a given value is multiplied by the same number (bigger than 1), it grows in size. In the Starter (page 77), the amount owing 'grows', to give this graph (see right).

What happens to this graph for negative values of x?
What is the value of 2^{-3}, for example?

We could either use a calculator: 2 x^y − 3 = (or 2 x^y 3 ± = if you are using an older model of calculator), which should give the result 2^-3 = 0.125 or $\frac{1}{8}$, or we could use properties we know from negative indices: $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$.



5.06 Decay curves

Exponential graphs (equation $y = a^x$) do not always show growth. If the base a is *less than* 1, the graph slopes downwards. Then the graph is called a **decay curve**.

Example

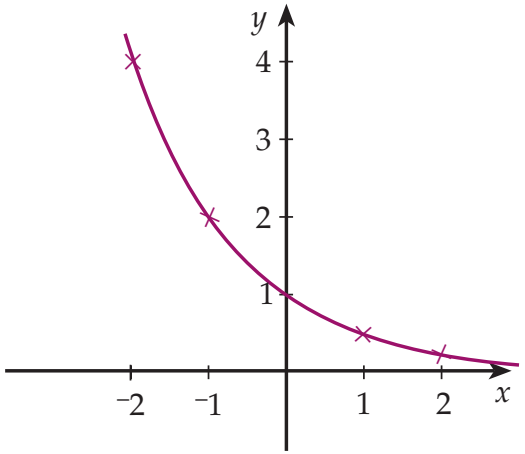
Draw the graph of $y = \left(\frac{1}{2}\right)^x$.

Answer

A calculator helps complete this table of values – work out powers of 0.5 rather than $\frac{1}{2}$.

x	$(0.5)^x$
$^-2$	4
$^-1$	2
0	1
1	0.5
2	0.25

Plot these points and join them up with a smooth curve.

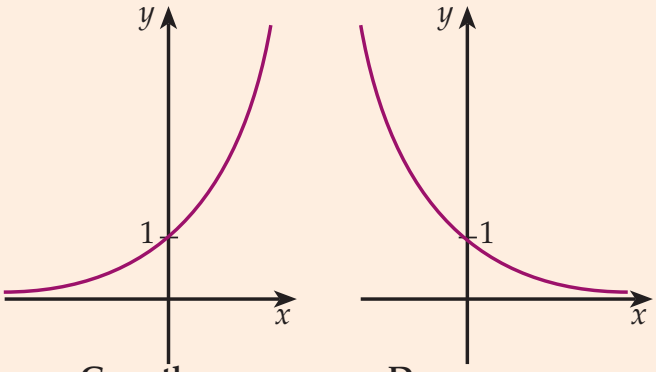


5.07 Key characteristics of exponential graphs

KEY POINTS

The general form of a simple exponential graph is $y = a^x$.

- If the base is greater than 1, we get a **growth** curve ($a > 1$); the graph slopes upwards everywhere. If the base is less than 1, we get a **decay** curve ($0 < a < 1$).



Growth curve $a > 1$ | **Decay curve** $0 < a < 1$

- The y-intercept is always 1. That is, exponential graphs always pass through the point (0, 1). This is because any number to the power of 0 gives a result of 1.
- The x-axis is an **asymptote**. On a growth curve for negative values of x, the graph becomes very close to the x-axis but never actually touches it.
- The base number a affects the steepness of the graph. For example, the graph of $y = 3^x$ is steeper than the graph of $y = 2^x$.

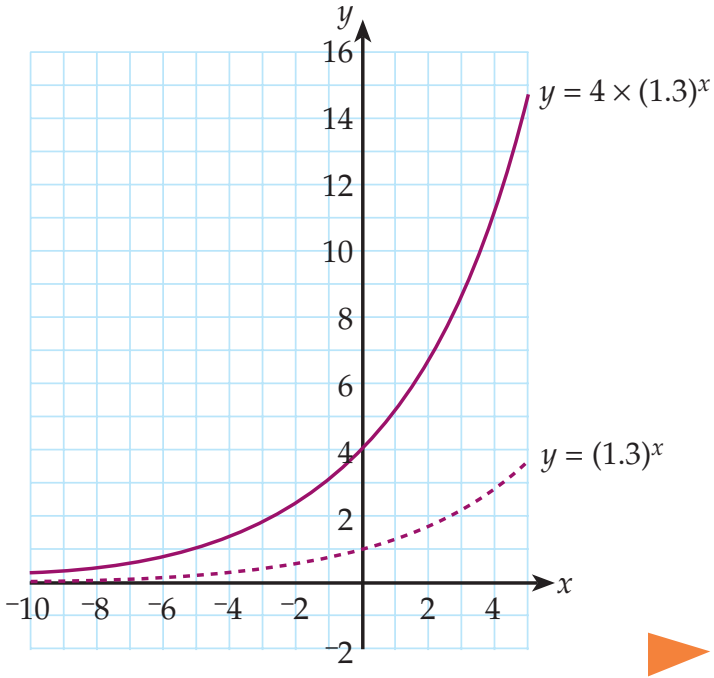
- An exponential function, of the form $y = a^x$, can take any x-value, but the y-value can only be a positive number. Technically, we say 'the **domain** of the function $y = a^x$ is the real numbers, and the **range** of the function $y = a^x$ is $\{y : y > 0, y \in \mathbb{R}\}$.'

5.08 Changes of scale – the form $y = b \times a^x$

Multiplying each y-value on the curve given by $y = a^x$ by the same constant alters the vertical scale. To transform the graph of $y = a^x$ to the graph of $y = b \times a^x$, scale the height of each point from the x-axis by a factor of b.

This is shown most clearly by the position of the y-intercept. On the graph of $y = a^x$, the intercept on the vertical axis is (0, 1). On the graph of $y = b \times a^x$, the vertical intercept is $(0, b)$.

For example, each point on the graph of $y = 4 \times (1.3)^x$ is four times the distance from the x-axis as the corresponding point on the graph of $y = (1.3)^x$.



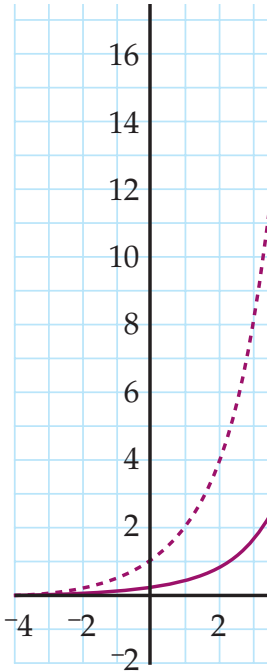
Two points on the graph of $y = (1.3)^x$ are (0, 1) and (1, 1.3). The points on the graph of $y = 4 \times (1.3)^x$ with the corresponding x-values are (0, 4) and (1, 5.2).

Example

Draw the graph of $y = \frac{1}{5} \times 2^x$.

Answer

Note: the graph of $y = 2^x$ is also shown for comparison.



5.09 Modelling – exponential graphs

Many real-life situations that involve growth and decay can be represented by an exponential model. These include compound interest, population growth and depreciation.

Example

Caitlin buys a new mobile phone. Its second-hand value (V), if she decided to sell it, can be represented by $V = 495 \times (0.6)^t$, where t is the number of years after buying the phone and V is the value of the phone in dollars.

a Calculate the vertical intercept of the graph and explain what it represents.

b Estimate how long it takes for the phone to halve in value. Explain by referring to the graph, and also support your answer with some calculations.

Answer

a The vertical intercept is calculated by substituting $t = 0$ into the equation $V = 495 \times (0.6)^t$:

$V = 495 \times (0.6)^0 = 495 \times 1 = 495.$

This value gives the information that the second-hand value immediately after purchase is \$495. Note: this is not necessarily what Caitlin paid for the phone new.

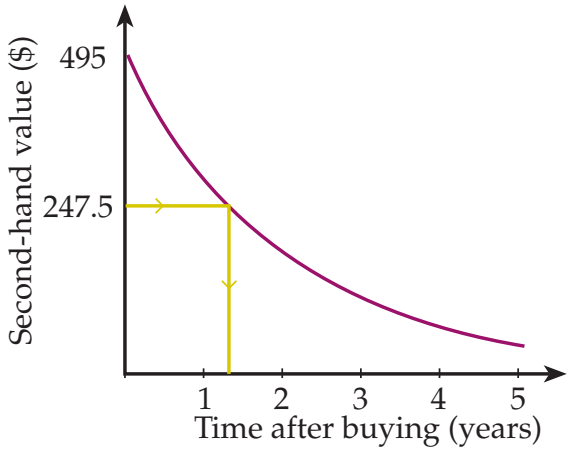
b To halve in value, the phone will be worth \$247.50.

The value of t that gives $V = 247.5$ will be somewhere between $t = 1$ and $t = 2$. From the graph, about $1\frac{1}{3}$ years would be a good estimate. We can confirm this estimate by calculating the value of V for $t = 1.2$ and $t = 1.5$.

For $t = 1.2$, $V = 495 \times (0.6)^{1.2} \approx 268$.

For $t = 1.5$, $V = 495 \times (0.6)^{1.5} \approx 230$.

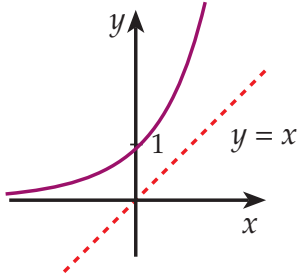
Given the downward-sloping property of the graph, it seems reasonable that the t-value that gives $V = 247.5$ would lie between 1.2 and 1.5.



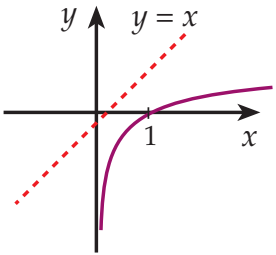
5.10 Logarithmic graphs

The function $y = \log_a(x)$ is the **inverse** of the function $y = a^x$. This means that the graph of $y = \log_a(x)$ is obtained by swapping x- and y-values from the $y = a^x$ graph. We reflect the exponential graph in the line $y = x$ to get the inverse graph.

Graph of $y = a^x$



Graph of $y = \log_a(x)$



Key characteristics of logarithmic graphs

KEY POINTS

- The y-axis is an asymptote.
- The graph passes through the point (1, 0). This is the x-intercept.
- The logarithmic function is only defined for positive values of x. This means a simple logarithmic graph only passes through points to the right of the y-axis. Technically, we say the **domain** of the function $y = \log_a(x)$ is the set $\{x: x > 0, x \in \mathbb{R}\}$.
- The logarithmic function can take on any y-value, positive or negative. Technically, we say the **range** of the function $y = \log_a(x)$ is the real numbers.

Initially, it is easiest to work with logarithms to base 10. Why? The log key on most calculators gives logarithms to base 10. Also, when using a spreadsheet, the =LOG() function calculates logarithms to base 10.

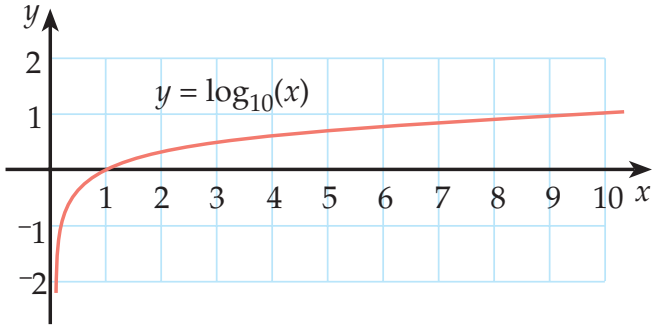
Example

Draw the graph of $y = \log_{10}(x)$.

Answer

Plot points:

x	y
0.01	$^{-}2$
0.1	$^{-}1$
1	0
10	1
100	2



5.11 Logarithmic functions where the base is not 10

In general, a logarithmic function is written as $y = \log_a(x)$. We will take a as being a counting number greater than 1. We can calculate logarithms to base a using at least three different methods.

1. On a spreadsheet, use the =LOG(number, base) function.
2. On a CAS calculator, use the log function.
3. Use the equivalent index equation and solve it.
4. Use the conversion formula $\log_a(x) = \dfrac{\log_{10}(x)}{\log_{10}(a)}$.

(Methods **3** and **4** are equivalent.)

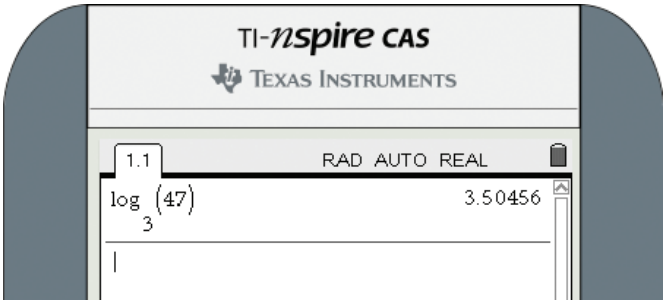
Example

Calculate $\log_3(47)$.

Answer

Method **1**: In a spreadsheet, type =LOG(47,3). This gives 3.504555375.

Method **2**: Use a CAS calculator.



Method **3**: Solve the equation $3^x = 47$.

$$3^x = 47$$
$$\log(3^x) = \log(47)$$
$$x\log(3) = \log(47)$$
$$x = \frac{\log(47)}{\log(3)} = 3.505 \text{ (4 sf)}$$

Method **4**:

$$\log_a(x) = \frac{\log_{10}(x)}{\log_{10}(a)}$$
$$\log_3(47) = \frac{\log_{10}(47)}{\log_{10}(3)}$$
$$= \frac{1.6721}{0.477\ 12}$$
$$= 3.504\ 555$$

To draw the graph of $y = \log_a(x)$, you could use a spreadsheet, or graphing software.

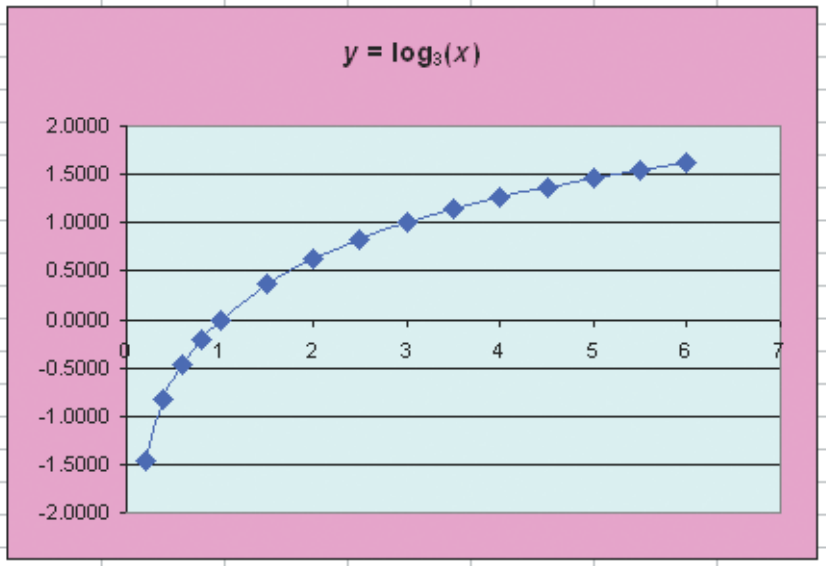
Example

Draw the graph of $y = \log_3(x)$.

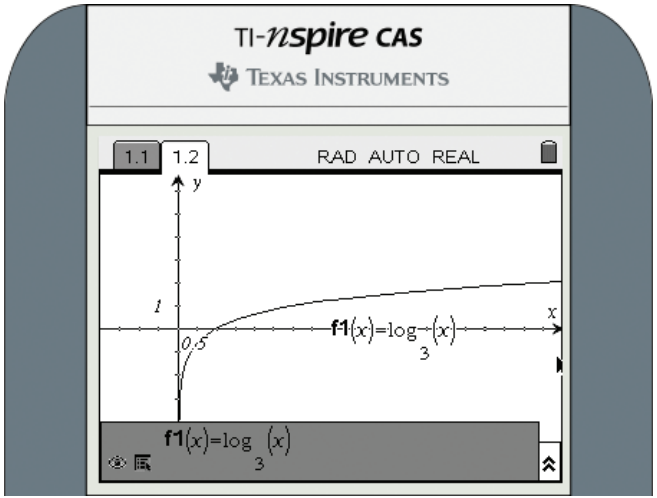
Answer

(From a spreadsheet.)

	A	B
1	x	$y = \log_3(x)$
2	0.2	-1.4650
3	0.4	-0.8340
4	0.6	-0.4650
5	0.8	-0.2031
6	1	0.0000
7	1.5	0.3691
8	2	0.6309
9	2.5	0.8340
10	3	1.0000
11	3.5	1.1403
12	4	1.2619
13	4.5	1.3691
14	5	1.4650
15	5.5	1.5517
16	6	1.6309
17		
18		



(From a CAS calculator.)



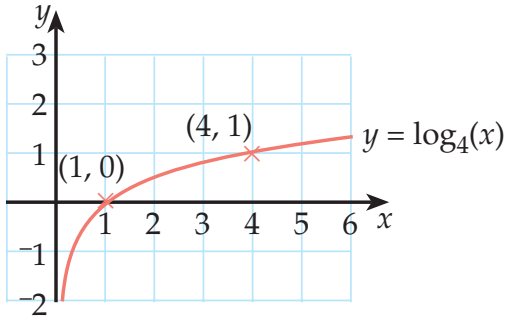
If accuracy is not important, and only the key features are required, you could draw the graph of $y = \log_a(x)$ as follows.

- The y-axis will be an asymptote.
- The graph passes through 1 on the x-axis – that is, the x-intercept is $(1, 0)$.
- The graph also passes through the point $(a, 1)$.

Example

Draw the graph of $\log_4(x)$.

Answer



6.01 Translations

Translations can be vertical, horizontal or a mixture of both.

Vertical translations

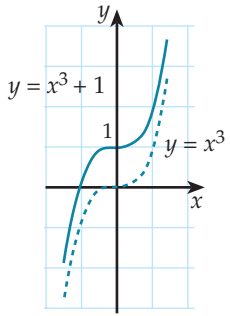
- To obtain the graph of $y = f(x) + a$, move the graph of $f(x)$ upwards by a units.
- To obtain the graph of $y = f(x) - a$, move the graph of $f(x)$ downwards by a units.

Example

Draw: **a** $y = x^3 + 1$ **b** $y = \log_3(x) - 2$.

Answer

a

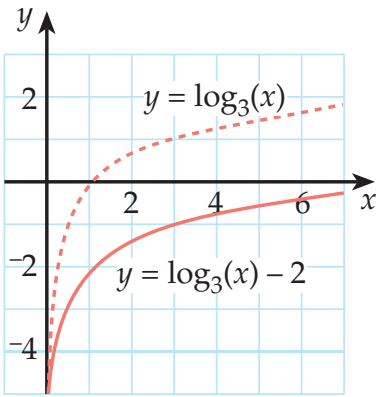


(The graph of $y = x^3$ is shown dashed as a comparison.)

Key features:

- the y-intercept is at (0, 1) – it has moved up by 1
- the domain remains the same – any x-value can be used, so the domain is the real numbers, $\mathbb{R}$
- the range is all real numbers, $\mathbb{R}$.

b



(The graph of $y = \log_3(x)$ is shown dashed as a comparison.)

Key features:

- the y-axis is an asymptote – moving the graph of $y = \log_3(x)$ up or down does not change the position of the asymptote
- the domain remains the same – any positive x-value can be used, so the domain is $x > 0$
- the range is all real numbers, $\mathbb{R}$.

Horizontal translations

- To obtain the graph of $y = f(x + b)$, move the graph of $f(x)$ to the left by b units.
- To obtain the graph of $y = f(x - b)$, move the graph of $f(x)$ to the right by b units.

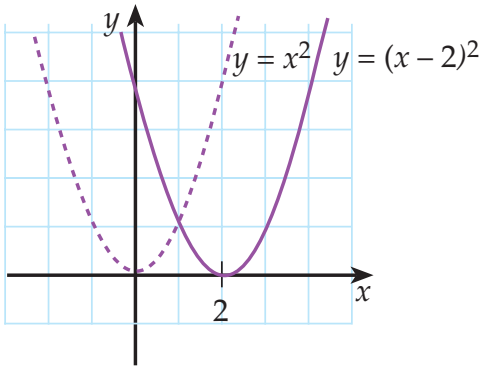
Example

Draw: **a** $y = (x - 2)^2$

b $y = 2^{x+1}$.

Answer

a

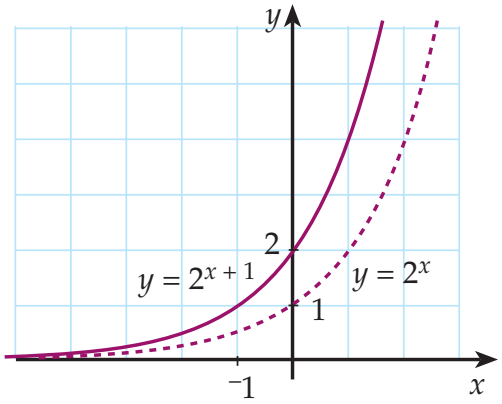


(The graph of $y = x^2$ is shown dashed as a comparison.)

Key features:

- the x-intercept is at (2, 0) – moving the graph to the right moves the x-intercept(s) to the right by the same distance
- the domain remains the same – any x-value can be used, so the domain is the real numbers, $\mathbb{R}$
- the range is all the non-negative real numbers, so the range is $y \geq 0$.

b



(The graph of $y = 2^x$ is shown dashed as a comparison.)

Key features:

- the x-axis is a horizontal asymptote
- the domain remains the same – any x-value can be used, so the domain is the real numbers, $\mathbb{R}$
- the range is all the positive real numbers, so is $y > 0$
- the vertical intercept is at (0, 2); this is found by substituting $x = 0$ into $y = 2^{x+1}$.

Horizontal translations can be confusing. The reason why the graph of $f(x)$ is moved right to get the graph of $f(x - a)$ is because if a were substituted, this would give $f(0)$.

For example, the parabola $y = (x - 2)^2$ has a root at $x = 2$. Thus, the bottom of the graph is at $x = 2$, not $^{-}2$.

Vertical and horizontal translations

When asked for a mixture of both vertical and horizontal translations, it is sometimes best to draw the basic function first.

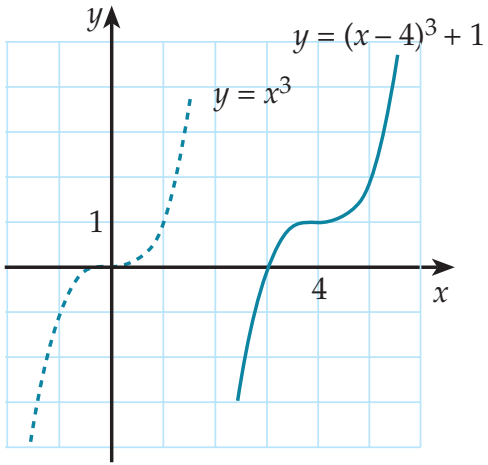
Example

Draw the graph of the function $y = (x - 4)^3 + 1$.

Answer

First draw the graph of $y = x^3$ (shown dashed in the diagram).

Then note that the $x - 4$ inside the brackets moves the graph 4 units to the right, and the $+ 1$ moves the graph up 1 unit.



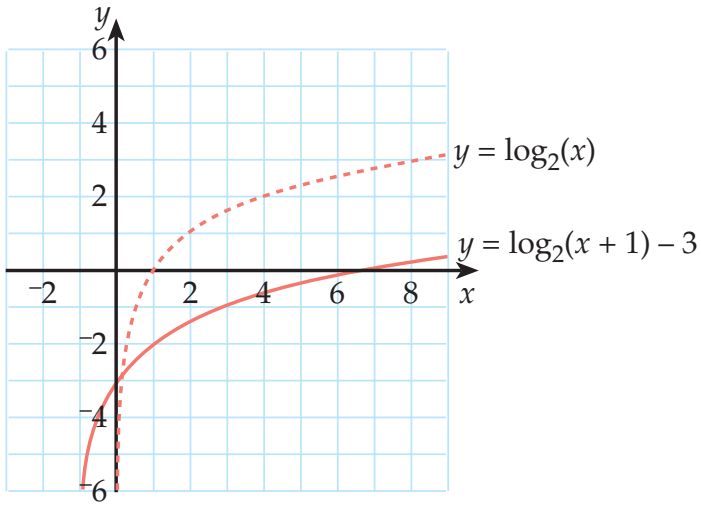
The asymptote translates in the same direction and by the same distance as the graph does.

Example

Draw the graph of $y = \log_2(x + 1) - 3$. Write the co-ordinates of the y-intercept and the equation of the vertical asymptote.

Answer

First draw the graph of $y = \log_2(x)$ (shown dashed in the diagram).



The $(x + 1)$ inside the logarithmic function moves the graph 1 unit to the left, and the -3 moves the graph down 3 units.

To calculate the y-intercept, substitute $x = 0$:

$$\begin{aligned} y &= \log_2(0 + 1) - 3 \\ &= \log_2(1) - 3 \\ &= 0 - 3 \\ &= {}^{-}3 \end{aligned}$$

The vertical asymptote is $x = {}^{-}1$.

6.02 Changes of scale

Changing the scale of a graph is equivalent to stretching the graph away at right-angles from an axis or compressing it towards an axis. When the scale is changed, the distance of a point from either the
x-axis or the y-axis is multiplied by a scale factor. There are two cases.

1 **Altering the vertical scale: $y = a \times f(x)$**
Here the constant a multiplies the result of the function, so the function is calculated first and then multiplied by a.

2 **Altering the horizontal scale: $y = f(ax)$**
Here the constant a is *inside* the function – the constant a multiplies x and then the function operates next.

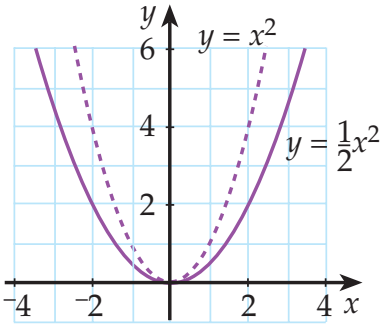
$a \times f(x)$

Here the constant alters the vertical scale. To obtain points on the graph of $y = a \times f(x)$, all heights above/below the x-axis on the graph of $y = f(x)$ are multiplied by a.

Example

Draw the graph of $y = \frac{1}{2}x^2$.

Answer

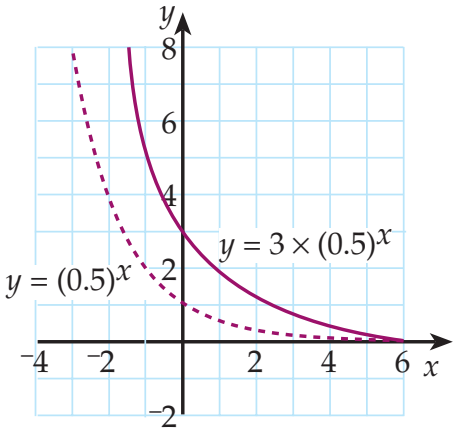


(The graph of $y = x^2$ is shown dashed as a comparison.)

Example

Draw the graph of $y = 3 \times (0.5)^x$. Write the equation of the asymptote and the co-ordinates of the y-intercept.

Answer



(The graph of $y = 0.5^x$ is shown dashed as a comparison.)

The line $y = 0$ is the horizontal asymptote.

Substitute $x = 0$ to calculate the y-intercept:

$y = 3 \times (0.5)^0 = 3 \times 1 = 3$

The y-intercept is (0, 3).

$f(ax)$

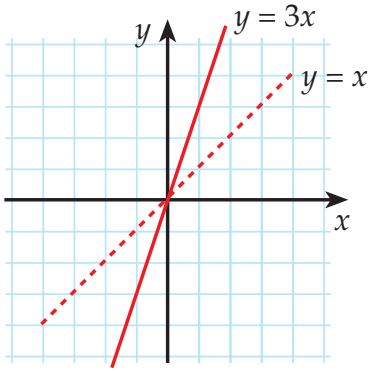
Here the constant alters the horizontal scale. Distances from the y-axis are reduced by a factor of a.

Example

Draw the graph of $y = 3x$.

Answer

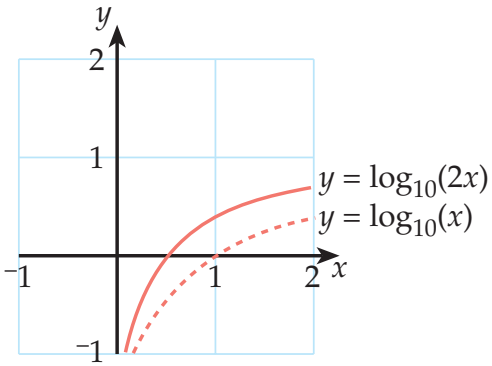
Note: all distances from the y-axis are reduced by a factor of 3. (The graph of $y = x$ is shown dashed as a comparison.)



Example

Draw the graph of $y = \log_{10}(2x)$.

Answer



Note: all distances from the y-axis are reduced by a factor of 2; that is, they are halved. (The graph of $y = \log_{10}(x)$ is shown dashed as a comparison.)

6.03 Reflections

The reflections here are of two types – either reflection in the x-axis, or reflection in the y-axis.

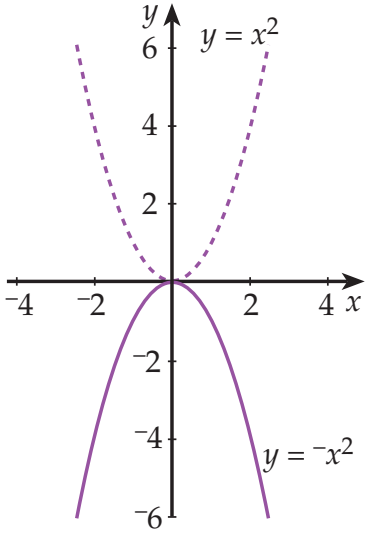
Reflection in the x-axis

The graph of $y = {}^{-}f(x)$ is obtained by reflecting the graph of $y = f(x)$ in the x-axis.

Example

Draw the graph of $y = {}^{-}x^2$.

Answer



(The graph of $y = x^2$ is shown dashed as a comparison.)

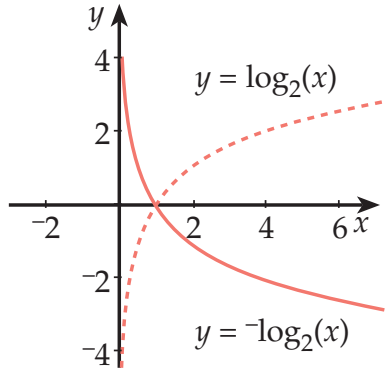
Key features:

- the domain remains the same – any x-value can be used, so the domain is the real numbers, $\mathbb{R}$
- the range is all the non-positive real numbers, so is $y \leq 0$.

Example

Draw the graph of $y = {}^{-}\log_2(x)$.

Answer



(The graph of $y = \log_2(x)$ is shown dashed as a comparison.)

Key features:

- the y-axis is still an asymptote
- the domain remains the same – any positive x-value can be used, so the domain is $x > 0$
- the range is all the real numbers, $\mathbb{R}$.

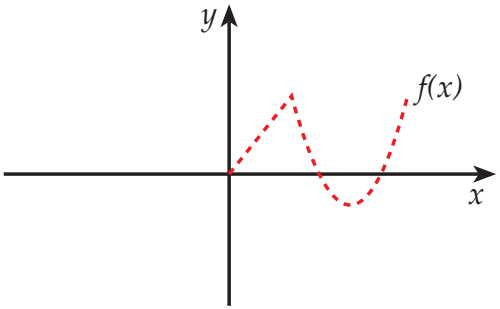
Reflection in the *y*-axis

The graph of $y = f(^{-}x)$ is obtained by reflecting the graph of $y = f(x)$ in the y-axis.

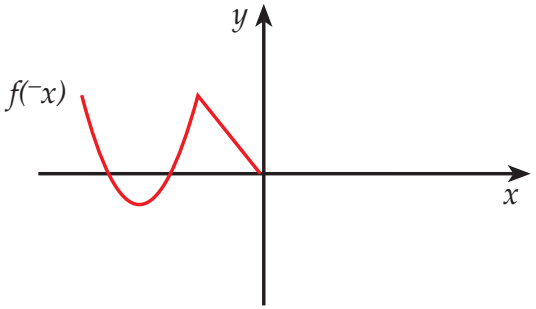
Example

The graph of $f(x)$ is shown.

Draw the graph of the function $f(^{-}x)$.



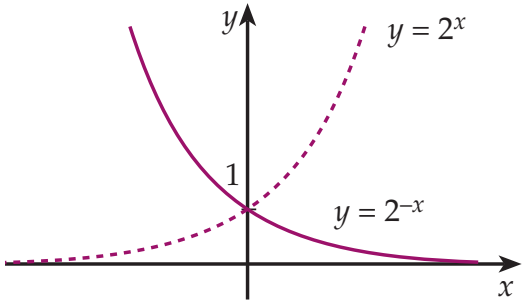
Answer



Example

Draw the graph of $y = 2^{-x}$.

Answer



(The graph of $y = 2^x$ is shown dashed as a comparison.)

Key features:

- the x-axis is still an asymptote
- the domain remains the same – any x-value can be used, so the domain is the real numbers, $\mathbb{R}$
- the range also remains the same – any positive value of y is possible, so the range is $y > 0$.

6.04 Several transformations at once

It is possible to have more than one kind of transformation occurring in any particular function. Two suggested approaches are:

1. technology
2. build up the final graph from a basic graph.

To draw the graph, you can use a graphics calculator, or graphing software, or a spreadsheet.

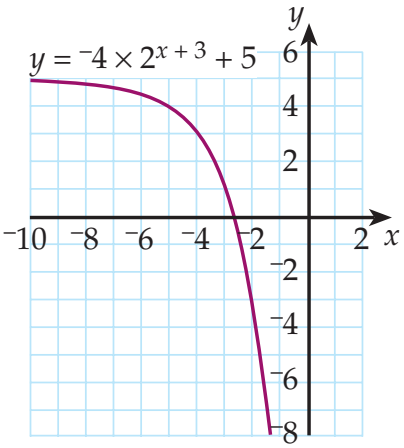
Example

Draw the graph of $y = {}^{-}4 \times 2^{x+3} + 5$.

Comment: This example includes horizontal and vertical translation, reflection, and change of scale.

Answer

1. Using graphing software – see graph at right.
2. If building up the graph from basic graphs, the best order to follow is the order of operations as they are applied to x.



The transformations in this graph can be explained in the following sequence.

i The basic graph is $y = 2^x$ and it passes through the points (0, 1) and (1, 2).

ii The '$x + 3$' moves the basic graph 3 units to the left.

iii The $^{-}4$ has two effects: multiplication by 4 increases all distances from the x-axis by 4, and the minus sign reflects the graph in the x-axis.

iv The '$+ 5$' moves the graph up 5 units.

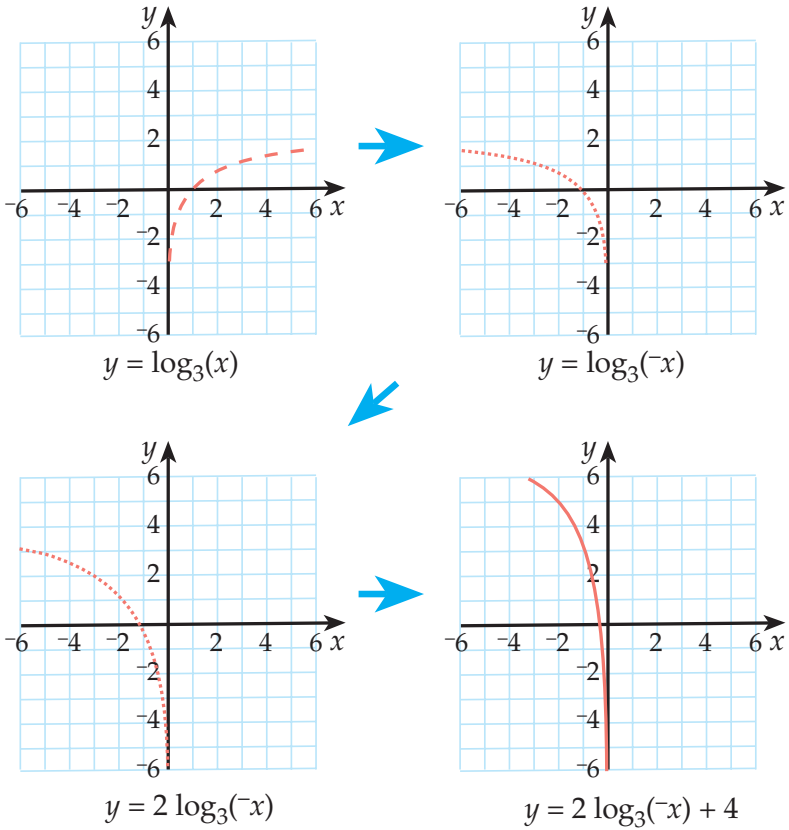
Example

Show how to build up the graph of $y = 2\log_3(^-x) + 4$ from the basic graph $y = \log_3(x)$.

Key features:

- the y-axis is an asymptote
- the domain is the negative real numbers, or $x < 0$
- the range is all the real numbers, $\mathbb{R}$.

Answer

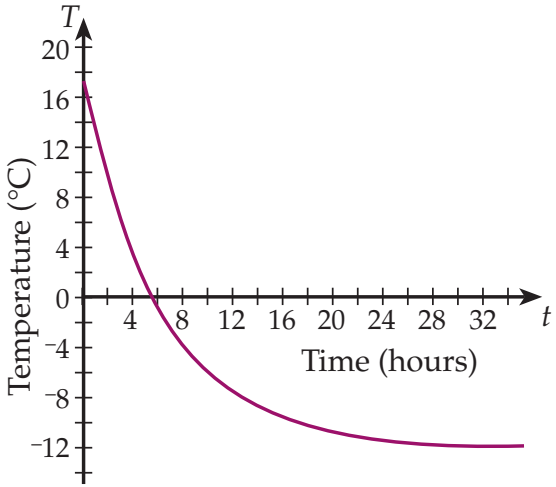


6.05 Interpreting features of models with exponential and logarithmic graphs

Real-life applications of exponential and logarithmic models include compound interest, population growth, decay, and so on. Sometimes, the model is translated up, down or sideways to allow for fixed values. The y-intercept usually gives information about initial conditions.

Example

A freezer is installed in a supermarket and switched on. The temperature (T) in degrees Celsius inside the freezer t hours after the freezer is switched on can be modelled by the equation $T = 30(0.85)^t - 12$.



a How does the graph show that the freezer cools fastest in the first hour?

b Calculate the value of the vertical intercept. Explain what this value represents in this model.

c Write the equation of the horizontal asymptote. Explain what it represents in this model.

d A similar freezer is installed in another supermarket. The temperature inside the supermarket is 21 °C and the freezer is set to cool to $^{-}9$ °C. The freezer lowers the temperature by 24% each hour. Express the relationship between the temperature of the freezer and the time elapsed as an exponential model.

e Discuss whether the model is realistic.

Answer

a The steepest part of the graph (between $t = 0$ and $t = 1$) is when the rate of cooling is highest.

b $T = 30(0.85)^0 - 12 = 30 \times 1 - 12 = 18$. The initial temperature of the freezer is 18 °C – this would be the same as the temperature inside the supermarket.

c The horizontal asymptote to the graph of $T = 30(0.85)^t - 12$ is the line $T = {}^{-}12$. The freezer will not cool to a temperature below $^{-}12$ °C, no matter how long it runs.

d If the freezer is lowering the temperature by 24% per hour, the base in the exponential function (or repeated multiplication factor) will be 0.76.

The equation will be of the form $T = A(0.76)^t - 9$.

Substitute $t = 0$ and $T = 21$ to calculate A:

$21 = A \times (0.76)^0 - 9$

$21 = A - 9$

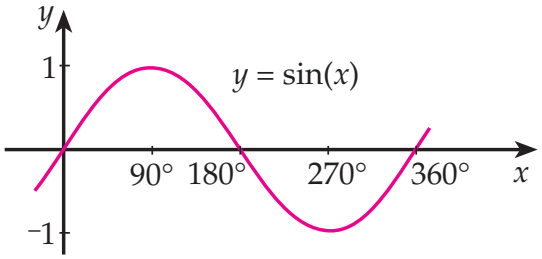
$A = 30$

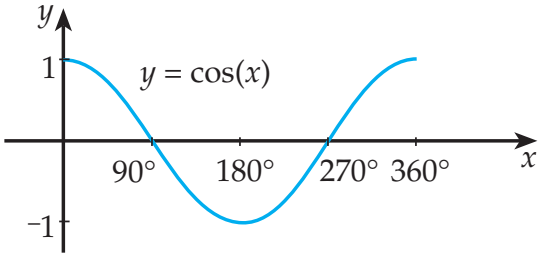
The equation is $T = 30(0.76)^t - 9$.

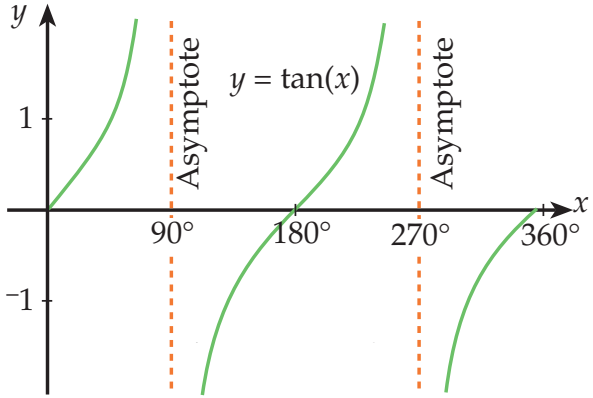
e The model is probably not realistic. The drop in temperature each hour is more likely to be constant as the freezer removes warm air, so a linear model, with a negative gradient, would be more realistic. The model also implies the freezer takes a very long time to reach its minimum operating temperature – it is more likely to reach this temperature and then stay there.

7.01 Introducing trig graphs

The three main trig graphs are those of $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$.







Note:

- The y-values for the sin and cos graphs always lie between 1 and $^{-}1$.
- $\tan(x)$ is undefined for the values 90° and 270°. The dashed lines on the diagram are called **asymptotes**. Here they are lines that the graph is very close to, but never actually touches.

7.02–7.04 Solving simple trig equations

There are several types of trig equation.

Trig equations of the form below usually have two solutions between 0° and 360°:

$\sin(x) = k$	or	$\cos(x) = k$	or	$\tan(x) = k$
for $^{-}1 \leq k \leq 1$		for $^{-}1 \leq k \leq 1$		for $k \in \mathbb{R}$

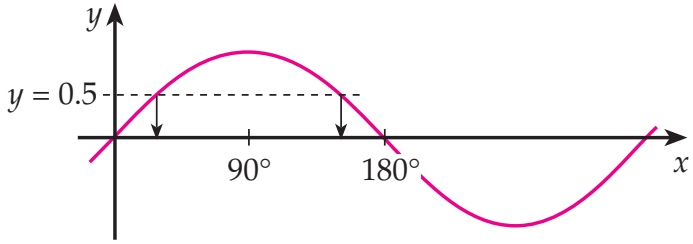
The simplest method to solve trig equations is to draw a graph, then use the symmetry properties of the graph to find the solutions.

Example

Solve the equation $\sin(x) = 0.5$, in degrees, for values of x between $0°$ and $360°$.

Answer

First, draw the graph of $y = \sin(x)$. On the same diagram, draw the horizontal line $y = 0.5$.



This line intersects the sin graph at two places. The x-values of these points are the two solutions we need.

To find the solution between $0°$ and $90°$, use a calculator to find $\sin^{-1}(0.5)$, which gives $30°$.

Now note that the sin graph is symmetrical about the line $x = 90°$. So, the other solution will be the same distance on the other side of this line, i.e. at $150°$.

The two solutions are $\{30°, 150°\}$.

It is a good idea to check each solution on a calculator. In this example, you would check by evaluating sin(30°) and sin(150°); each gives 0.5.

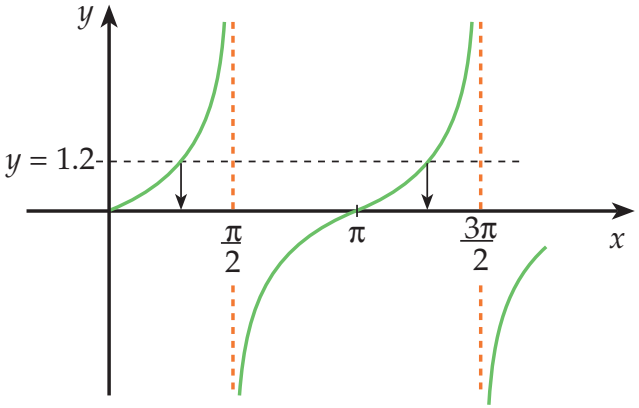
The above example used symmetry in a vertical line. It is also possible to use the periodic property of trig graphs.

Example

Solve the equation $\tan(x) = 1.2$ for $0 \leq x \leq 2\pi$.

Answer

First, draw the graph of $y = \tan(x)$ and, on the same diagram, draw the horizontal line $y = 1.2$.



This line intersects the tan graph at two places. The x-values of these points are the two solutions we need.

To find the solution between 0 and $\frac{\pi}{2}$, use a calculator to find $\tan^{-1}(1.2)$, which gives 0.876 radians.

Now note that the tan graph repeats itself every π units (i.e. 3.142, the value of π). The other solution must be $0.876 + 3.142 = 4.018$ radians.

The two solutions are {0.876, 4.018} (in radians).

The $0 \leq x \leq 2\pi$ part after the equation indicates that the solution is to be given in radians.

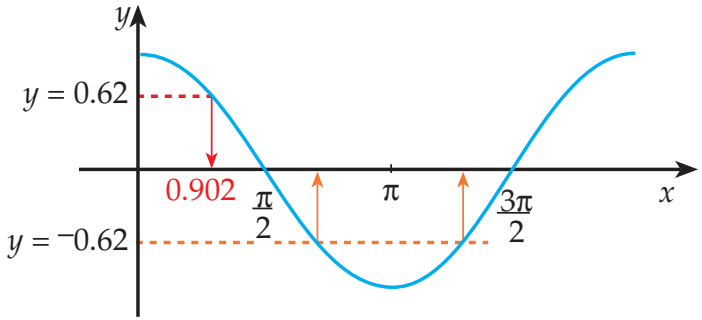
A further type of problem uses rotational symmetry.

Example

Solve the equation $\cos(x) = {}^{-}0.62$ for $0 \leq x \leq 2\pi$.

Answer

Draw the cos graph and the line $y = {}^{-}0.62$ on the same diagram.



Note that there are two solutions: one is between $\frac{\pi}{2}$ and π, and the other is between π and $\frac{3\pi}{2}$.

The cos graph has rotational symmetry about $\left(\frac{\pi}{2}, 0\right)$.

Thus, first find $\cos^{-1}(0.62)$ which gives 0.902 radians. By rotational symmetry, the angle that gives $^{-}0.62$ must be the same distance from π as 0.902 is from 0. That is, the answer is $\pi - 0.902$ or $3.142 - 0.902 = 2.24$ radians.

The other answer is the same distance on the other side of π,
i.e. $3.142 + 0.902 = 4.04$.

The solutions are {2.24, 4.04} (in radians).

7.05 & 7.06 Rearranging simple trig equations

The same rules we use in algebra for rearranging linear equations also work for trig equations.

Example

Solve the equation $12\cos(x) = 7$ for $0° \leq x \leq 360°$.

Answer

Rearrange to obtain $\cos(x) = \frac{7}{12} = 0.5833$.

Recall that cos is positive for angles between 0° and 90°, and for angles between 270° and 360°. This tells us where to look for the two solutions.

$\cos^{-1}(0.5833) = 54.3°$

One solution is 54.3°.

The other solution, between 270° and 360°, is $360° - 54.3° = 305.7°$.

The solution set is {54.3°, 305.7°}.

Example

Solve $2\sin(\theta) - \sqrt{3} = 0$ for $0 \leq \theta \leq 2\pi$.

Answer

Rearrange to get $2\sin(\theta) = \sqrt{3}$:

$\sin(\theta) = \dfrac{\sqrt{3}}{2} = 0.8660$

$\sin^{-1}$ on a calculator gives $\theta = 60°$ or $\dfrac{\pi}{3}$ radians.

This is one solution. The value of sin here is positive so, by symmetry, the other solution is between 90° and 180°:

$180° - 60° = 120° = 120 \times \dfrac{\pi}{180} = \dfrac{2\pi}{3}$.

Example

Solve $\tan^2(x) = 16$ for $0 \leq x \leq 2\pi$.

Answer

$\tan^2(x) = 16$ so $\tan(x) = 4$ or $\tan(x) = {}^{-}4$.

Solve each equation separately.

$\tan(x) = 4$ has solutions between $0°$ and $90°$, and between $180°$ and $270°$.

Hence, $\tan^{-1}(4) = 1.326$ gives the first solution.

The solution between $180°$ and $270°$ is $\pi + 1.326 = 3.142 + 1.326 = 4.467$.

(Note: in this solution, the full calculator values have been used.)

$\tan(x) = {}^{-}4$ has solutions between $90°$ and $180°$, and between $270°$ and $360°$.

Note: tan is negative for these ranges of values.

The solution between $90°$ and $180°$ is $\pi - 1.326 = 1.816$.

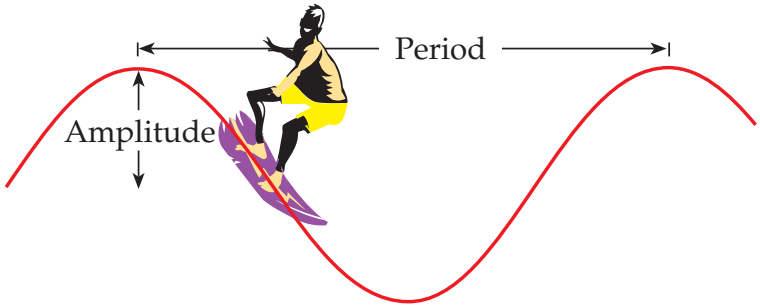
The solution between $270°$ and $360°$ is $2\pi - 1.326 = 4.957$.

The full solution set is $\{1.326, 1.816, 4.467, 4.957\}$.

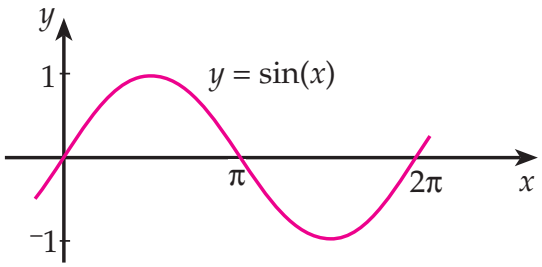
7.07 Applications of trig graphs

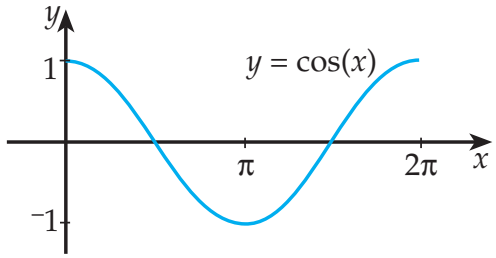
The shape of sin and cos curves can be described using two pieces of information – the **period** and the **amplitude**.

- Each of the sin, cos and tan functions is **periodic** – that is, they repeat their values every so often as we move along the x-axis. For example, $\sin(30°) = \sin(390°) = \sin(750°)$, etc. The period of a trig function tells us how often the graph repeats. The period is the minimum positive sideways movement needed to map the graph onto itself. For both sin and cos, the period $= 2\pi = 360°$.



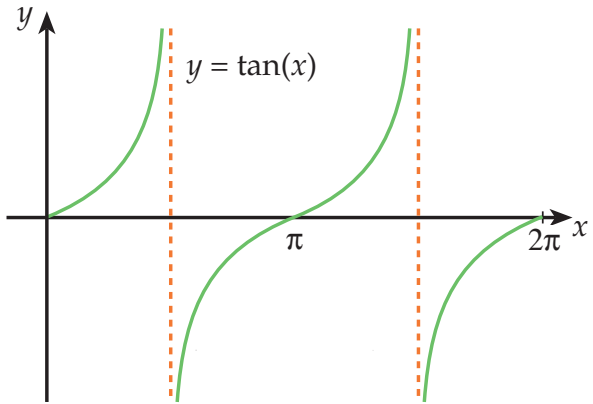
- The **amplitude** is *half* the difference between the largest y-value and the smallest y-value. For a trig graph where the x-axis goes 'through the middle', the amplitude is the height of the graph above the x-axis. The graphs of sin and cos have an amplitude of 1.





One application of both sin and cos functions is that they mathematically model wave motion. This is useful in electricity (alternating current), physics (e.g. radio waves) and oceanography. The **period** describes the distance between successive waves, and the **amplitude** describes the height of each wave.

Note that the tan graph has asymptotes at odd multiples of 90°.



7.08 & 7.09 Transformations of the sin and cos graphs

In functions such as $a\sin(x)$, $\sin(bx)$, $\sin(x + c)$ and $\sin(x) + d$, the constants a, b, c and d transform the graph in certain ways.

$a\sin(x)$ and $a\cos(x)$

Multiplying sin or cos by a number alters the **amplitude** of the graph.

Example

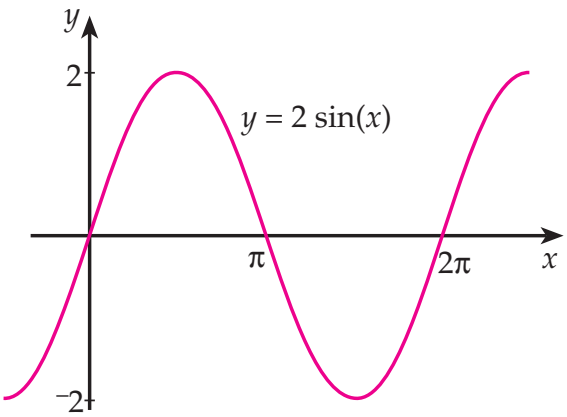
Sketch the graphs of:

a $y = 2\sin(x)$

b $y = \frac{1}{2}\cos(x)$.

Answer

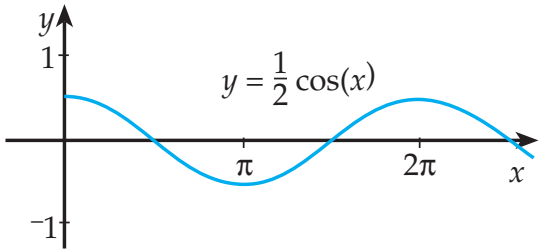
a



Key features of $y = 2\sin(x)$ and its **graph**:

- the **amplitude** is 2 (half the difference between the largest y-value and the smallest y-value)
- the **period** is 2π
- the **domain** is $\mathbb{R}$
- the **range** is $-2 \leq y \leq 2$.

b



Key features of $y = \frac{1}{2}\cos(x)$ and its **graph**:

- the **amplitude** is $\frac{1}{2}$ (half the difference between the largest y-value and the smallest y-value)
- the **period** is 2π
- the **domain** is $\mathbb{R}$
- the **range** is $\frac{-1}{2} \leq y \leq \frac{1}{2}$.

sin(*bx*) and cos(*bx*)

If the angle is multiplied by a constant *before* the sin or cos is taken, then the frequency changes accordingly. For example, $\sin(2x)$ repeats itself twice between 0° and 360° instead of only once, and therefore the **period** is halved.

Example

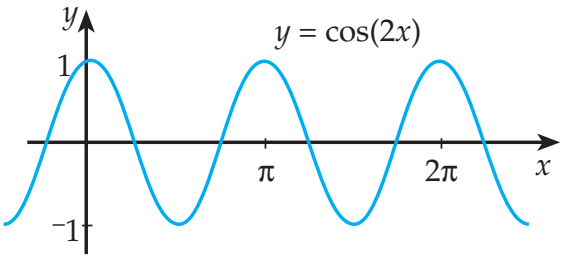
Sketch the graphs of:

a $y = \cos(2x)$

b $y = \sin(3x)$.

Answer

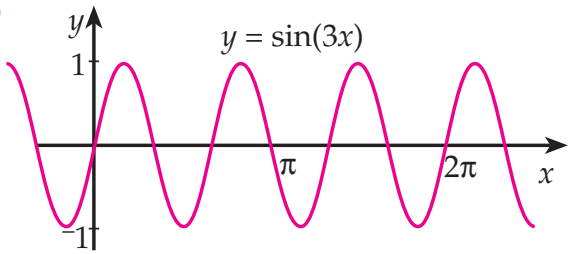
a



Key features of $y = \cos(2x)$ and its **graph**:

- the **amplitude** is 1
- the **period** is π
- the **domain** is $\mathbb{R}$
- the **range** is $^{-}1 \leq y \leq 1$.

b



Key features of $y = \sin(3x)$ and its **graph**:

- the **amplitude** is 1
- the **period** is $\frac{2\pi}{3}$ or $120°$
- the **domain** is $\mathbb{R}$
- the **range** is $^{-}1 \leq y \leq 1$.

$\sin(x \pm c)$ and $\cos(x \pm c)$

Adding (or subtracting) a constant to the angle *before* the sin or cos is taken moves the graph sideways. If the constant is positive, then:

- **adding** a constant moves the graph to the **left**
- **subtracting** a constant moves the graph to the **right**.

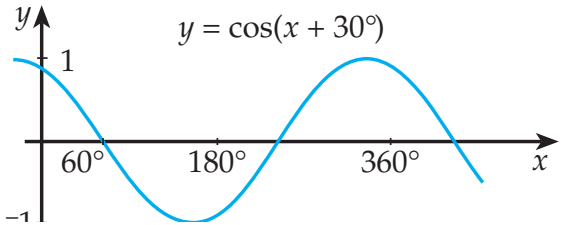
Example

Sketch the graphs of:

a $y = \cos(x + 30°)$ **b** $y = \sin(x - 45°)$.

Answer

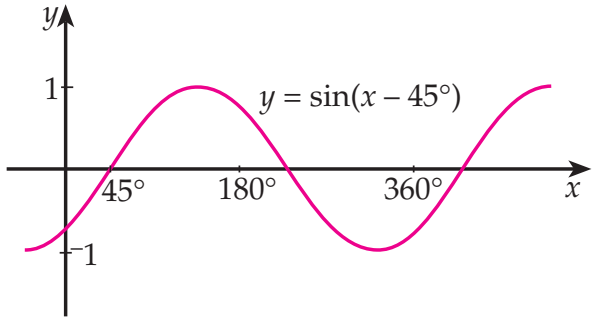
a



Key features of $y = \cos(x + 30°)$ and its **graph**:

- the **amplitude** is 1
- the **period** is 2π or $360°$
- the **domain** is $\mathbb{R}$
- the **range** is $^{-}1 \leq y \leq 1$.

b



Key features of $y = \sin(x - 45°)$ and its **graph**:

- the **amplitude** is 1
- the **period** is 2π or $360°$
- the **domain** is $\mathbb{R}$
- the **range** is $^{-}1 \leq y \leq 1$.

$\sin(x) + d$ and $\cos(x) + d$

Adding a constant to the *result* of a sin or cos function moves the graph *up* or *down*.

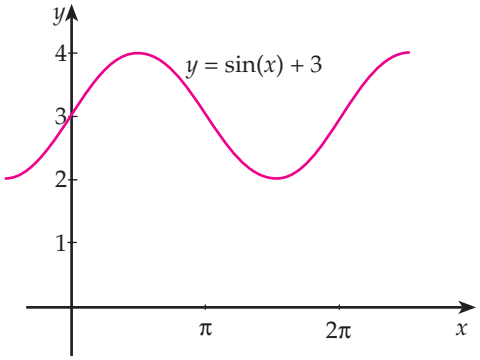
Example

Sketch the graphs of:

a $y = \sin(x) + 3$ **b** $y = \cos(x) - 1$.

Answer

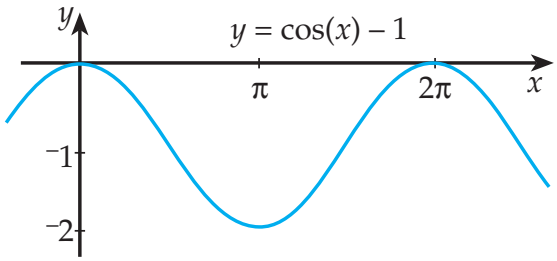
a



Key features of $y = \sin(x) + 3$ and its **graph**:

- the **amplitude** is 1; this is half the difference between the largest y-value (4) and the smallest y-value (2)
- the **period** is 2π or $360°$
- the **domain** is $\mathbb{R}$
- the **range** is $2 \leq y \leq 4$.

b



Key features of $y = \cos(x) - 1$ and its **graph**:

- the **amplitude** is 1; this is half the difference between the largest y-value (0) and the smallest y-value ($^{-}2$)
- the **period** is 2π or $360°$
- the **domain** is $\mathbb{R}$
- the **range** is $^{-}2 \leq y \leq 0$.

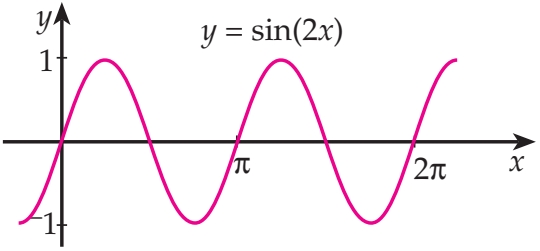
Example

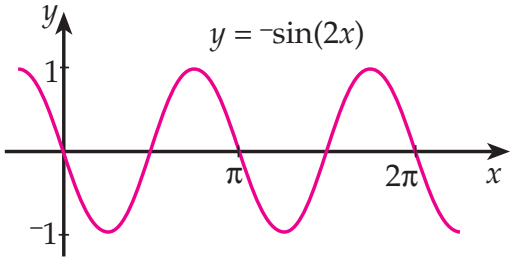
Sketch the graph of $y = 4 - \sin(2x)$.

Answer

It is often best to sketch this kind of example in stages (as shown by the three graphs below). Note that $4 - \sin(2x)$ is the same as $^{-}\sin(2x) + 4$.

$\sin(2x)$ repeats itself twice between 0° and 360°. Multiplying by $^{-}1$ changes the sign of every y-value, i.e. reflects the graph in the x-axis.

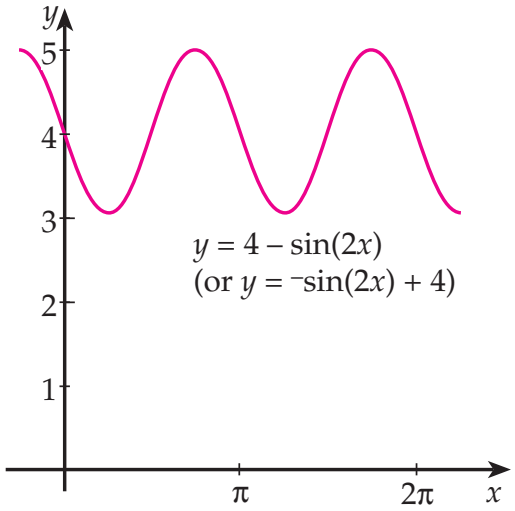




Finally, adding 4 moves the graph up 4 units.

Key features of $y = 4 - \sin(2x)$ and its **graph**:

- the **amplitude** is 1; this is half the difference between the largest y-value (5) and the smallest y-value (3)
- the **period** is π or 180°
- the **domain** is $\mathbb{R}$
- the **range** is $3 \leq y \leq 5$.



7.10 Equations resulting from translations of trig graphs: $\sin(x \pm a) = b$ and $\cos(x \pm a) = b$

Degrees

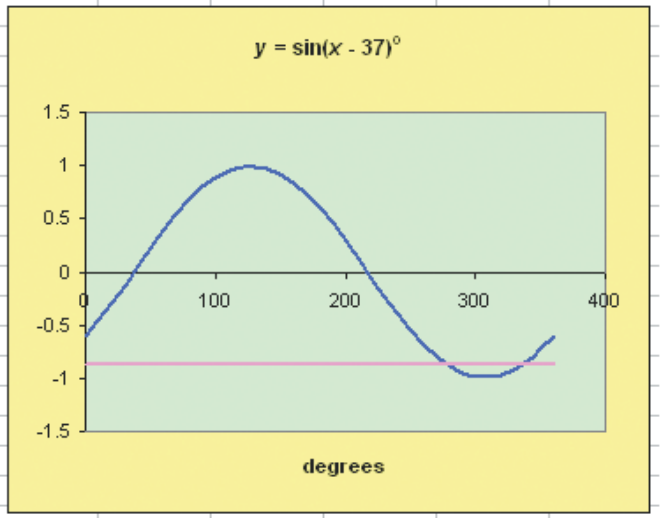
This example is the same as the next example, which is solved using more traditional methods.

Example

Use a spreadsheet to picture the solution of the equation $\sin(x - 37°) = {}^{-}0.87$, for $0° \leq x \leq 360°$. Estimate the solution(s).

Answer

	A	B	C
1	*x* in degrees	value of function	constant
2			
3	0	-0.60182	-0.87
4	20	-0.29237	-0.87
5	40	0.052336	-0.87
6	60	0.390731	-0.87
7	80	0.681998	-0.87
8	100	0.891007	-0.87
9	120	0.992546	-0.87
10	140	0.97437	-0.87
11	160	0.838671	-0.87
12	180	0.601815	-0.87
13	200	0.292372	-0.87
14	220	-0.05234	-0.87
15	240	-0.39073	-0.87
16	260	-0.682	-0.87
17	280	-0.89101	-0.87
18	300	-0.99255	-0.87
19	320	-0.97437	-0.87
20	340	-0.83867	-0.87
21	360	-0.60182	-0.87



The sin graph takes a y-value of $^{-}0.87$ for values of x near 280° and 340°. Reasonable estimates for the solution of the equation are 280° and 340°.

Example

Solve $\sin(x - 37°) = {}^{-}0.87$ for $0° \leq x \leq 360°$.

Answer

There are two angles between 0° and 360° that have a sin value of ${}^{-}0.87$. These angles are 240.5° and 299.5°.

$(x - 37°)$ has two possible values: 240.5° and 299.5°.

Therefore:

$$x - 37° = 240.5° \qquad \text{or} \qquad x - 37° = 299.5°$$
$$x = 240.5° + 37° \qquad x = 299.5° + 37°$$
$$= 277.5° \qquad = 336.5°$$

The symmetry of the sin graph means we can get these two angles by calculating $\sin^{-1}(0.87)$ to obtain 60.5°, and then adding 60.5° to 180° and subtracting 60.5° from 360°. This gives the angles 240.5° and 299.5°.

Radians – multiples of π

Example

Solve $\cos\left(x - \frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$ for $0 \leq x \leq 2\pi$.

Answer

Note that $\frac{1}{\sqrt{2}} = 0.7071$ and $\cos^{-1}(0.7071)$ gives 45° exactly when full calculator values are used. Another possible angle is 315° or $^{-}45°$. Converting to radians, these angles are $\frac{\pi}{4}$ and $\frac{^{-}\pi}{4}$ respectively.

$$x - \frac{\pi}{3} = \frac{\pi}{4} \quad \text{or} \quad x - \frac{\pi}{3} = \frac{^{-}\pi}{4}$$

$$x = \frac{\pi}{4} + \frac{\pi}{3} \qquad x = \frac{^{-}\pi}{4} + \frac{\pi}{3}$$

$$= \frac{7\pi}{12} \qquad = \frac{\pi}{12}$$

Alternatively, the problem could be performed in degrees, converting to radians at the final stage.

Radians in decimal form

Some problems will require angles to be used from outside the usual 0° to 360° or 0 to 2π range. Here is an example.

Example

Solve the equation $\sin(x - 1.9) = {}^{-}0.88$ for $0 \leq x \leq 2\pi$.

Answer

An angle with a sin value of ${}^{-}0.88$ has two solutions, 4.22 and 5.21:

$x - 1.9 = 4.22$ or $x - 1.9 = 5.21$

$x = 6.12$ $x = 7.11$

Now $x = 7.11$ is outside the range $0 \leq x \leq 2\pi$.

Subtracting 2π gives us exactly the same angle, but within this range:
i.e. $x = 7.11 - 2\pi = 0.82$.

The two solutions are 0.82 and 6.12.

7.11 & 7.12 Changing the frequency: $\sin(ax) = b$ and $\cos(ax) = b$

The number a in this type of equation controls the **frequency** of the graph. This means there could be more than just two solutions between 0 and 2π.

If $a = 3$, say, there will be six solutions. This is because the graph repeats itself three times between 0 and 2π, instead of just once – so there will be three times as many solutions.

Example

Solve $\cos(3x) = 0.75$ for $0° \leq x \leq 360°$.

Answer

The equation $\cos(x) = 0.75$ has two solutions (in degrees) between $0°$ and $360°$. These are $41.4°$ and $318.6°$.

Add $360°$ to each of these, and further solutions would be $401.4°$ and $678.6°$.

Add another $360°$ to obtain more solutions: $761.4°$ and $1038.6°$.
(Note: the reason for adding $360°$ is that we will be dividing by 3, which will bring these angles back to ones less than $360°$.)

$3x = 41.4°$ or $318.6°$ or $401.4°$ or $678.6°$ or $761.4°$ or $1038.6°$

$x = 13.8°$ or $106.2°$ or $133.8°$ or $226.2°$ or $253.8°$ or $346.2°$.

We can use graph-drawing software to picture the solution to a trig equation.

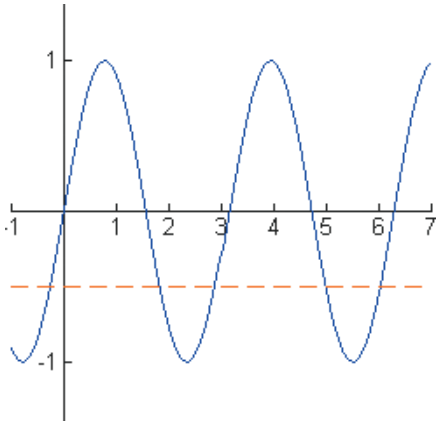
The example below is the same as the next example (to the right), which is solved using more traditional methods.

Example

Produce a picture of the solution to the equation $\sin(2x) = \frac{-1}{2}$ for $0 \leq x \leq 2\pi$.

Estimate the solution(s).

Answer



This graph was drawn using TI InterActive™.

Note:

1. The values are calculated using radians, because radians are implied by the restriction 'for $0 \leq x \leq 2\pi$'.
2. The sin graph repeats itself twice between 0 and 2π. This means there are four solutions.
3. The sin graph takes a y-value of $\frac{-1}{2}$ for values of x near 1.8, 2.8, 5 and 6. These four values are reasonable estimates for the solution of the equation.

Example

Solve $\sin(2x) = \frac{-1}{2}$ for $0 \leq x \leq 2\pi$.

Answer

The equation $\sin(2x) = \frac{-1}{2}$ has two solutions (in degrees) between $0°$ and $360°$. These are $210°$ and $330°$.

Add $360°$ to each of these, and further solutions would be $570°$ and $690°$. (Later we will be dividing by 2, which will bring these angles back to ones less than $360°$.)

$2x = 210°$ or $330°$ or $570°$ or $690°$

$x = 105°$ or $165°$ or $285°$ or $345°$.

In radians:

$$\frac{105\pi}{180} \quad \text{or} \quad \frac{165\pi}{180} \quad \text{or} \quad \frac{285\pi}{180} \quad \text{or} \quad \frac{345\pi}{180}$$

gives

$$\frac{7\pi}{12} \quad \text{or} \quad \frac{11\pi}{12} \quad \text{or} \quad \frac{19\pi}{12} \quad \text{or} \quad \frac{23\pi}{12}.$$

Note: these solutions in radians can either be given as multiples of π (see above) or changed to ordinary numbers. As ordinary numbers, the four solutions are: 1.833, 2.880, 4.974 and 6.021. These correspond closely to the estimates 1.8, 2.8, 5 and 6 given by the graph-drawing software (refer to the graph at left).

8.01 Piecewise functions

Some functions are defined by more than one rule. The definition of the function depends on the value of x.

For example, a telephone account could be charged at a flat rate of \$30 for the first 100 calls per month, and then at 20 cents for each extra call. The piecewise function for the cost to the subscriber per total number of calls would be:

$$f(x) = \begin{cases} 30, & 0 \leq x \leq 100,\ x \in \mathbb{W} \\ 30 + 0.2(x - 100), & x > 100,\ x \in \mathbb{W} \end{cases}$$

When evaluating the function for different values of x, the first decision to make is which rule to use – this is apparent from the range of x-values that is given after each rule.

A **piecewise function** has different rules for different values of x. The functions are called piecewise because they are described in *pieces*!

Example

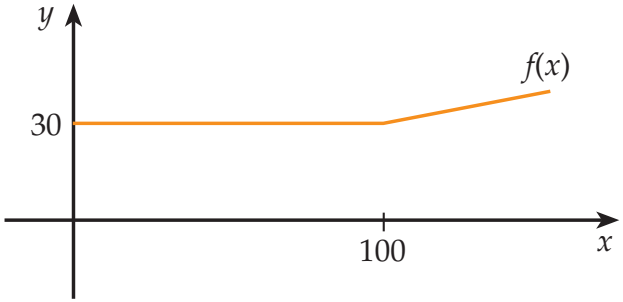
Using the above example of the telephone account, find the cost to the subscriber of 125 calls per month – i.e. find $f(125)$.

Answer

First, decide which rule to use. 125 is greater than 100, so use the second formula:

$$\begin{aligned} f(x) &= 30 + 0.2(x - 100) \\ f(125) &= 30 + 0.2(125 - 100) \\ &= 30 + 5 \\ &= \$35 \end{aligned}$$

The graph of this function is shown here.



Note: this graph has been drawn as though the domain were $\mathbb{R}$ rather than $\mathbb{W}$.

When drawing a piecewise function, only draw the parts of the graph corresponding to the restrictions on x.

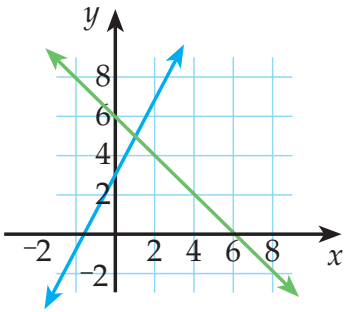
Example

Draw the graph of the piecewise function defined as follows:

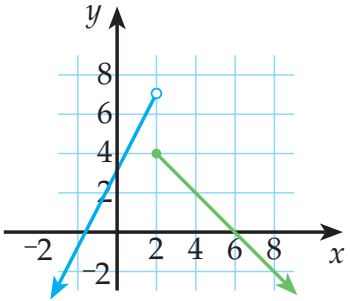
$$f(x) = \begin{cases} 2x + 3, & x < 2 \\ 6 - x, & x \geq 2 \end{cases}$$

Answer

If we draw the graphs for both rules, with no restrictions, we have the following diagram:



However, we want only the part of the first graph (shown in blue) to the left of $x = 2$, and we only want the part of the second graph (shown in green) to the right of $x = 2$:



Notice that we show the point (2, 7) is not included (it is not part of the graph) by using an open circle. The point (2, 4) is part of the graph, and we show that by using a closed (filled-in) circle.

The absolute value function is an example of a piecewise function. We can express this function using the notation $y = |x|$ but this notation disguises the fact that the absolute function is really a piecewise function with these two rules:

$$f(x) = \begin{cases} x, & x \geq 0, x \in \mathbb{R} \\ -x, & x < 0, x \in \mathbb{R} \end{cases}$$

8.02 Non-linear piecewise functions

The rule(s) for one or more parts of a piecewise function do not have to always be linear (with straight-line graphs). They can include any of the other graphs in this course, including cubics, rational functions and exponential curves.

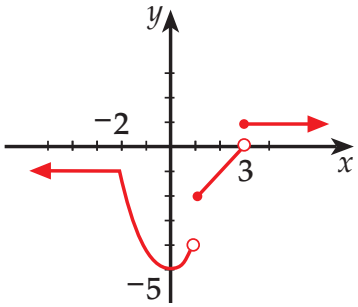
Example

Draw the graph of $f(x)$ given below.

$$f(x) = \begin{cases} -1, & x < -2 \\ x^2 - 5, & -2 \leq x < 1 \\ x - 3, & 1 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

Answer

Consider each rule separately:



There are actually four different rules here but only the portions corresponding with the given intervals are drawn. For example, the line $y = x - 3$ is shown only for values of x between 1 and 3. The usual convention of representing included points with a closed circle, and excluded points with an empty circle, is followed. At a point, such as $(^{-}2, ^{-}1)$, that is common to both adjacent intervals, no circle is necessary. The arrows on the two lines indicate that the lines extend indefinitely.

9.01 Right-angled triangles

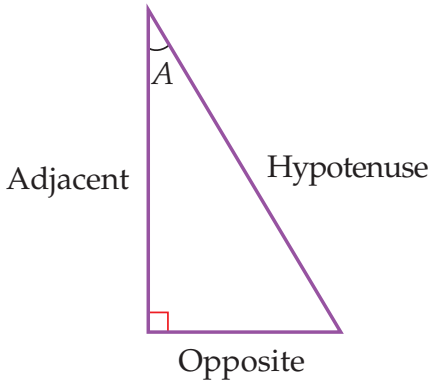
Trigonometry is first introduced by working with right-angled triangles, and looking at the relationships between the lengths of sides and the sizes of angles.

If one particular angle is marked as angle A, then we can refer to the three sides as follows.

- The **hypotenuse** is the longest side of the right-angled triangle; it is always opposite the right-angle.
- The **adjacent** side is adjacent to, or next to, the marked angle and joins the marked angle to the right-angle.
- The **opposite** side is opposite the marked angle.

We use these symbols for the names of the sides:

- h = hypotenuse
- a = adjacent side
- o = opposite side.



The three formulae that link any pair of sides with the marked angle are:

1 $\sin(A) = \frac{o}{h}$ or $o = h \times \sin(A)$

2 $\cos(A) = \frac{a}{h}$ or $a = h \times \cos(A)$

3 $\tan(A) = \frac{o}{a}$ or $o = a \times \tan(A)$

sin, cos and tan are the three 'trig' formulae used with right-angled triangles. Their full names are **sine**, **cosine** and **tangent**. Their values are usually worked out on a calculator.

Calculating the length of a side

In any right-angle trigonometry question where you have to calculate the length of a side, follow these steps.

1. Identify the two sides that are given or marked.
2. Choose between the sin, cos or tan formula, and write it down.
3. Substitute in the values from the diagram.
4. Calculate the required side.

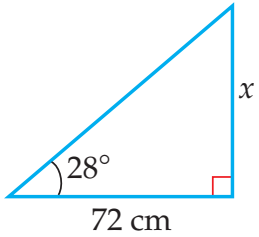
In the trig examples here, we will give answers rounded to 4 sf (or 1 dp in the case of angles) and will take the given measurements as being exact. Later, however, in application-type examples, the best approach is to round appropriately.

Example

Calculate the side marked x.

Answer

Use the tan formula because x is the opposite side and the 72 cm side is the adjacent side.



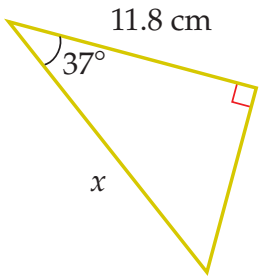
$$\tan(A) = \frac{o}{a} \text{ or } o = a \times \tan(A)$$

$$x = 72 \times \tan(28°)$$

$$= 38.28 \text{ cm (4 sf)}$$

Example

Calculate the side marked x.



Answer

Use the cos formula because x is the hypotenuse and the 11.8 cm side is the adjacent side.

$$\cos(A) = \frac{a}{h} \quad \text{or} \quad a = h \times \cos(A)$$

$$11.8 = x \times \cos(37°)$$

$$x = \frac{11.8}{\cos(37°)}$$

$$= 14.78 \text{ cm (4 sf)}$$

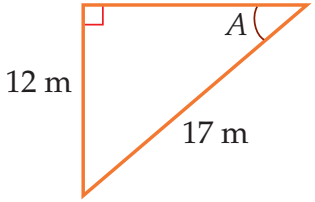
Calculating an angle given two sides

Trigonometry can also be used to calculate the sizes of angles in right-angled triangles when we know the lengths of two of the sides.

1. Identify the two sides that are given in relation to the marked angle, A.
2. Choose between the sin, cos or tan formula, and write it down.
3. Substitute in the values from the diagram.
4. Calculate the required angle. This involves changing a fraction to a decimal and then using one of the *inverse trig* keys ($\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$) on a calculator.

Example

Calculate the angle marked A.



Answer

Use the sin formula because the 12 m side is the opposite side and the 17 m side is the hypotenuse.

$$\frac{o}{h} = \sin(A)$$

$$\sin(A) = \frac{12}{17} = 0.7059$$

$$A = 44.9° \text{ (1 dp)}$$

9.02 Applications of right-angled triangles

Right-angle trigonometry has many applications:

- navigation
- building
- surveying
- civil engineering.

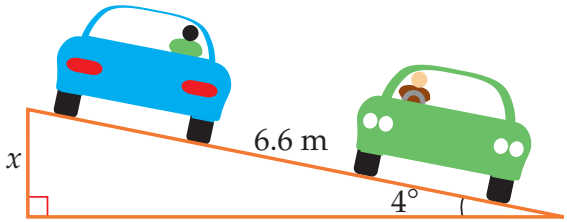
Example

When roads go round bends, engineers design them with a 'camber'. The camber makes it easier for cars to stay on the road when cornering. At one place on a curve, a road has a camber of 4° and is 6.6 m across. Calculate the difference in height between the sides of the road.

Answer

There is enough information in the diagram to draw a right-angled triangle. The difference in heights is marked x. Use the sin formula.

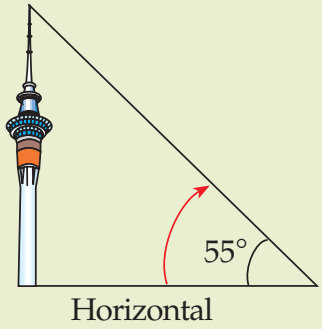
$o = h \times \sin(A)$
$x = 6.6 \times \sin(4°)$
$= 0.46$ m (2 sf)



9.03 Angles of elevation and depression

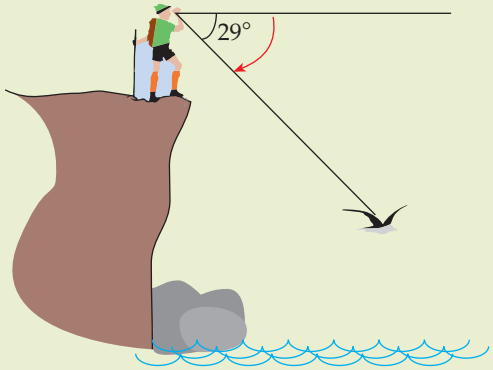
Surveying often involves calculations with angles measured in relation to the horizontal.

Angles of **elevation** are measured *upwards* from the horizontal.



The angle of elevation of the top of this tower from the ground is 55°.

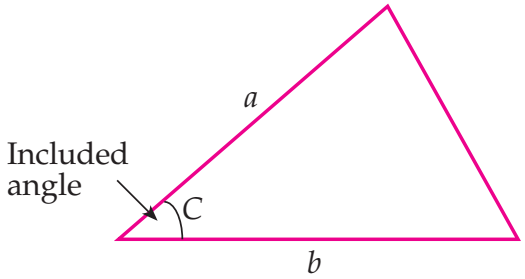
Angles of **depression** are measured *downwards* from the horizontal.



The angle of depression of this seagull from the observer at the top of a cliff is 29°.

9.04 Area of a triangle – two sides and the 'included' angle

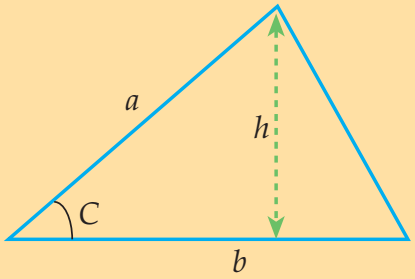
We can use trigonometry to calculate the area of a triangle if we know the lengths of *two* sides and the size of the angle between the two sides. This angle is often referred to as the **included** angle.



This ‘proof’ shows how we develop a formula for this calculation.

a and b are the two *given* sides. We use C as the label for the *included* angle.

We add a line to the triangle to show the ‘height’, and label this h.



From trigonometry, $h = a \times \sin(C)$.

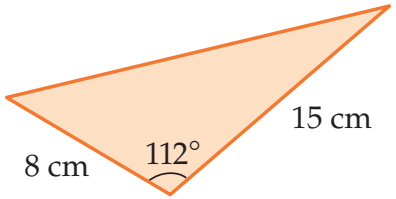
The area of the triangle is given by:

$$\begin{aligned}\text{area} &= \frac{1}{2}\,\text{base} \times \text{height} \\ &= \frac{1}{2} \times b \times h \\ &= \frac{1}{2} \times b \times a \times \sin(C) \\ &= \frac{1}{2}\,ab\sin(C).\end{aligned}$$

This formula is often useful because the measurements given do not have to be perpendicular. This is unlike the triangle formula learned earlier, which only applies if b and h are at right-angles.

Example

Calculate the area of this triangle.



Answer

$$\begin{aligned}\text{Area} &= \frac{1}{2}ab\sin(C)\\ &= \frac{1}{2}\times 8\times 15\times \sin(112°)\\ &= 60\times \sin(112°)\\ &= 55.63\ \text{cm}^2 \quad (4\text{ sf}) \quad \text{(from calculator)}\end{aligned}$$

9.05 Simple trig identities

An **identity** is an equation that is true for all possible values of the variables in the equation.

Most equations in x are only satisfied by some values of x. For example, $x^2 - 5x + 6 = 0$ is made true by the values 2 and 3, and $\sin(x) = 0$ is satisfied by angles such as $0°$, $180°$, $360°$, …, etc. (i.e. multiples of $180°$).

However, identities are true for *all* values of x. Sometimes, an 'equals' symbol with three bars (like this: $\equiv$) is used for identities, instead of the standard equals symbol with two bars (like this: $=$).

Examples

$3(x - 4) \equiv 3x - 12$

$\sin^2(x) + \cos^2(x) \equiv 1$

The $\equiv$ symbol means 'identically equal to' or 'equal for all values of the variable(s)'.

Both of these results are true for all values of x. These equations are two examples of an identity.

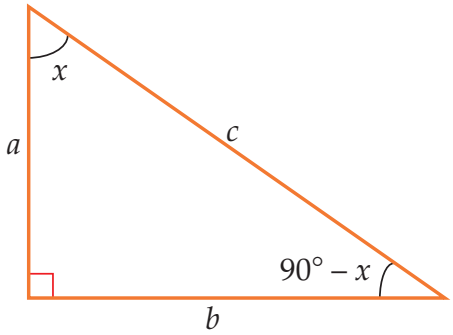
Three simple trig identities

There are three simple trig identities that are easy to demonstrate or prove using basic trigonometry. Most other simple trig identities can be derived from these three basic ones.

1 $\sin(x) \equiv \cos(90° - x)$
$\cos(x) \equiv \sin(90° - x)$

Proof

Consider the right-angled triangle below.



If one angle is x, then the complementary angle must be $(90° - x)$.

$$\sin(x) = \frac{b}{c}$$

$$\cos(90° - x) = \frac{b}{c}$$

Therefore, $\sin(x) = \cos(90° - x)$.

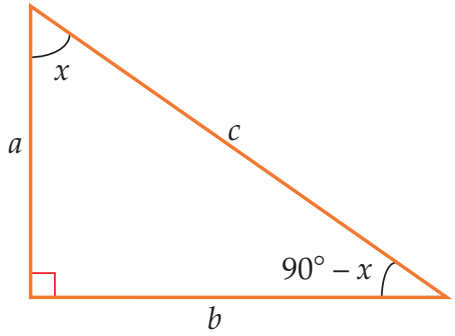
Similarly, $\cos(x) = \frac{a}{c} = \sin(90° - x)$.

2 $\dfrac{\sin(x)}{\cos(x)} \equiv \tan(x)$

This identity holds only for values of x in the domain of $\tan(x)$. It does not hold for values such as $\frac{\pi}{2}$, $\frac{3\pi}{2}$, etc. (or 90°, 270°, etc.) because the cos of these values is 0, and so the fraction $\dfrac{\sin(x)}{\cos(x)}$ is undefined.

Proof

Consider the diagram:



$$\frac{\sin(x)}{\cos(x)} = \frac{b}{c} \div \frac{a}{c} = \frac{b}{c} \times \frac{c}{a} = \frac{b}{a}$$

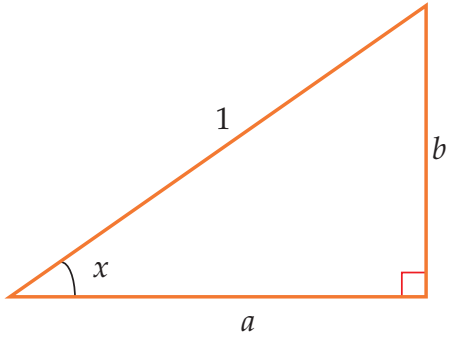
But $\tan(x) = \dfrac{b}{a}$ (by definition).

Hence, $\dfrac{\sin(x)}{\cos(x)} = \tan(x)$.

3 $\sin^2(x) + \cos^2(x) \equiv 1$

Proof

Consider this triangle. The hypotenuse is 1 unit long.



$$\sin(x) = \frac{b}{1} = b$$

$$\cos(x) = \frac{a}{1} = a$$

Pythagoras for this triangle gives $a^2 + b^2 = 1^2 = 1$.
But $a^2 = \sin^2(x)$, and $b^2 = \cos^2(x)$.
Therefore, $\sin^2(x) + \cos^2(x) = a^2 + b^2 = 1$.

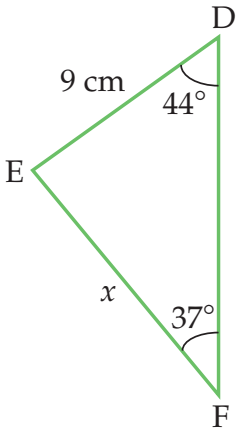
Notice how we write $\sin(x) \times \sin(x)$ as $\sin^2(x)$. The expression $\sin^2(x)$ shows it is the sin that is being squared, not the angle. This expression is more elegant than representing $\sin(x) \times \sin(x)$ as $(\sin(x))^2$ because fewer brackets are used, and it also avoids potential confusion with $\sin(x^2)$, which has a very different meaning.

10.01 & 10.02 Calculating sides with the sine rule

Example

Write the sine rule for this triangle, and use it to calculate the length marked x.

Answer



Each side and *opposite* angle are matched in the sine rule equation:

$$\frac{d}{\sin(D)} = \frac{e}{\sin(E)} = \frac{f}{\sin(F)}.$$

Substitute the known lengths and angles from the diagram:

$$\frac{x}{\sin(44°)} = \frac{9}{\sin(37°)}$$

$$x = \frac{9 \times \sin(44°)}{\sin(37°)} \quad \text{(multiplying both sides by } \sin(44°)\text{)}$$

$$= 10.39 \text{ cm} \quad (4 \text{ sf})$$

Use a calculator for this kind of question, but do not round any values until the end of the calculation.

In most sine rule problems, one of the three fractions in the rule is not needed to solve the problem. It can be ignored.

$$\frac{d}{\sin(D)} = \cancel{\frac{e}{\sin(E)}} = \frac{f}{\sin(F)}$$

10.03 & 10.04 Calculating angles with the sine rule

The sine rule can be used to work out angle sizes in triangles if sufficient information is given.

The most convenient way of working with the sine rule when calculating angles is to write the fractions upside-down:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

The first example shows how to make $\sin(\theta)$ the subject of a sine rule type equation.

Example

Make $\sin(\theta)$ the subject of the equation $\frac{\sin(56°)}{4} = \frac{\sin(\theta)}{9}$.

Answer

Multiply both sides of the equation by 9:

$$\sin(\theta) = \frac{9 \times \sin(56°)}{4}$$

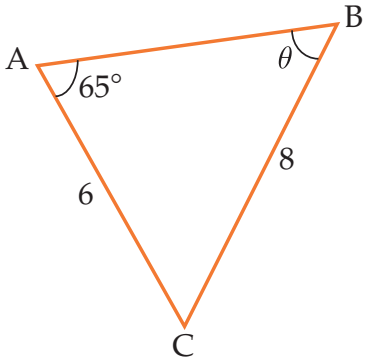
The next example shows how to calculate the size of an angle using the sine rule.

Example

Calculate the size of angle θ.

Answer

Write the sine rule, and substitute the given values from the diagram:



$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

$$\frac{\sin(\theta)}{6} = \frac{\sin(65°)}{8}$$

$$\sin(\theta) = \frac{6 \times \sin(65°)}{8}$$

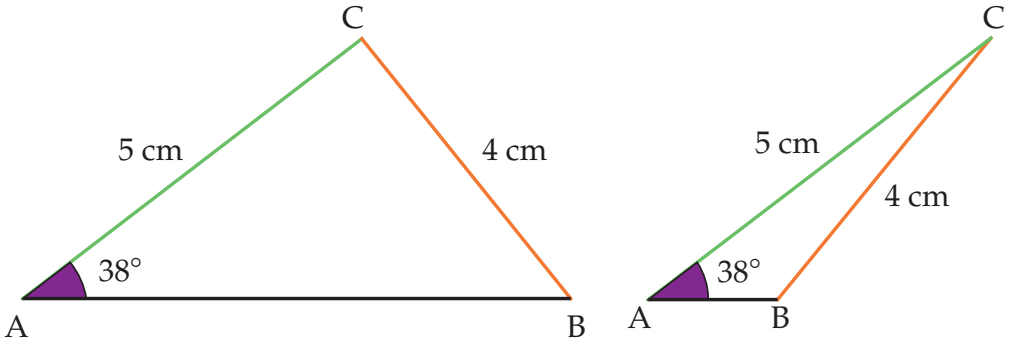
$$= 0.6797$$

$$\theta = 42.8° \text{ (1 dp)} \quad \text{(using the } \sin^{-1} \text{ key on a calculator)}$$

The last parts of the working are done on a calculator. Although you can write rounded values as part of the working, it is best not to actually do any rounding of calculator values until the end.

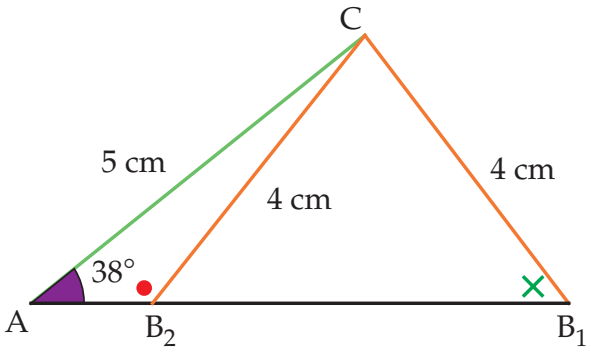
10.06 The ambiguous case

Sometimes, it is possible to draw two *different* triangles that have the same three measurements in common. For example, both of these triangles have the measurements $A = 38°$, $a = 4$ cm and $b = 5$ cm.



The word **ambiguous** means 'capable of two meanings'. If we work out angle B, there are two possible answers – angle B can either be acute or obtuse.

What is the relationship between the two possible angles at B? In the diagram at right, these angles are at positions B_1 and B_2.



$\angle AB_2C + \angle CB_2B_1 = 180°$ ($\angle$'s on line add to 180°)

$\angle CB_2B_1 = \angle CB_1B_2$ (isos Δ, base $\angle$'s =)

Therefore, the two marked angles • and × ($\angle AB_2C$ and $\angle CB_1B_2$) are **supplementary** (add to 180°).

We can see how the sine rule can give two answers for angle B in this example by looking at the working.

Example

Calculate the two values of angle B for these triangle measurements.
$A = 38°$, $a = 4$ cm and $b = 5$ cm.

Answer

Use the sine rule:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

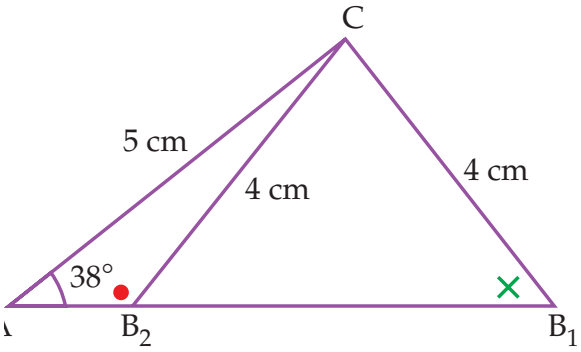
$$\frac{\sin(38°)}{4} = \frac{\sin(B)}{5}$$

$$\sin(B) = \frac{5 \times \sin(38°)}{4}$$

$$= 0.7696$$

$$B = 50.3° \text{ or } 129.7°$$

Note: 129.7° is the supplement of 50.3°.



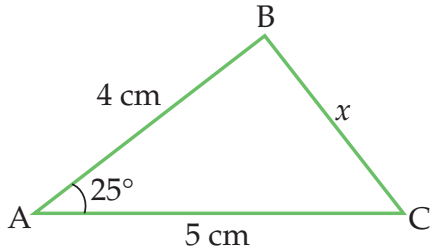
The **ambiguous case** occurs when the *shorter* of the two given sides is opposite the given angle.

11.01 & 11.02 Calculating sides with the cosine rule

The cosine rule is used to calculate the third side of a triangle when we are given measurements for two sides and the *included* angle (the angle in between). If the cosine rule is expressed as $a^2 = b^2 + c^2 - 2bc\cos(A)$ then the given sides would be b and c, and the given (included) angle would be A. The side to be calculated, if expressed in this form, would be a.

Example

Use the cosine rule to calculate the length of the side marked x in this triangle.



Answer

$a^2 = b^2 + c^2 - 2bc\cos(A)$ (cosine rule)

$x^2 = 5^2 + 4^2 - 2 \times 5 \times 4 \times \cos(25°)$

$= 25 + 16 - 40\cos(25°)$

$= 41 - 40\cos(25°)$

$= 4.748$

$x = \sqrt{4.748} = 2.179$ cm (4 sf)

1 Remember to multiply the cos value by the number in front before doing the subtraction.

2 Make sure you calculate the square root at the final stage.

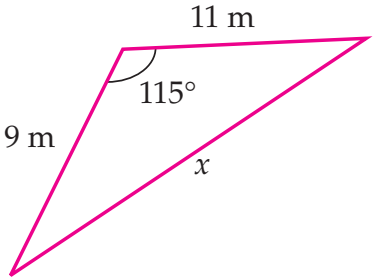
Labelling

When the triangle is not already labelled with letters, you can either:

- add your own labels and then follow the cosine rule, or
- substitute directly, using the fact that the side you are required to calculate is *opposite* the given angle.

Example

Use the cosine rule to calculate x.



Answer

$$\begin{aligned} x^2 &= 9^2 + 11^2 - 2 \times 9 \times 11 \times \cos(115°) \\ &= 81 + 121 - 198\cos(115°) \\ &= 202 - 198\cos(115°) \\ &= 285.7 \\ x &= \sqrt{285.7} = 16.9 \text{ m (3 sf)} \end{aligned}$$

The example shows two important points.

1 Check that you can get this answer on your calculator. When the given angle is obtuse, it has a negative cosine value. In the working, what happens is this:

$$\begin{aligned} 202 - 198 \times \cos(115°) &= 202 - 198 \times {}^{-}0.4226 \\ &= 202 - {}^{-}83.7 \\ &= 202 + 83.7 \\ &= 285.7 \end{aligned}$$

2 The two given sides (b and c) are interchangeable in the cosine rule formula. This is because the expression $b^2 + c^2 - 2bc\cos(A)$ has the same value as $c^2 + b^2 - 2cb\cos(A)$. So, in this example, another way of substituting the given sides would be $11^2 + 9^2 - 2 \times 11 \times 9 \times \cos(115°)$. This would give the same final answer.

11.03 & 11.04 Calculating angles with the cosine rule

If the lengths of all three sides of a triangle are given, we can use the cosine rule to calculate the size of any angle in the triangle. This involves rearranging the cosine rule formula to make $\cos(A)$ the subject:

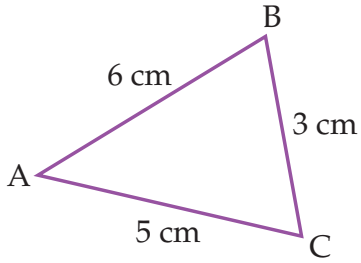
$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos(A) \\ 2bc\cos(A) &= b^2 + c^2 - a^2 \\ \cos(A) &= \frac{b^2 + c^2 - a^2}{2bc} \end{aligned}$$

In this formula, the side *opposite* the angle being calculated is the one that is subtracted. For example, in the triangle DEF, the cosine rule for calculating $\angle F$ would be:

$$\cos(F) = \frac{d^2 + e^2 - f^2}{2de}.$$

Example

Use the cosine rule to calculate the size of $\angle A$ in this triangle.



Answer

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{5^2 + 6^2 - 3^2}{2 \times 5 \times 6}$$

$$= \frac{25 + 36 - 9}{60}$$

$$= \frac{52}{60} = 0.8\dot{6}$$

$$A = 29.9° \text{ (1 dp)}$$

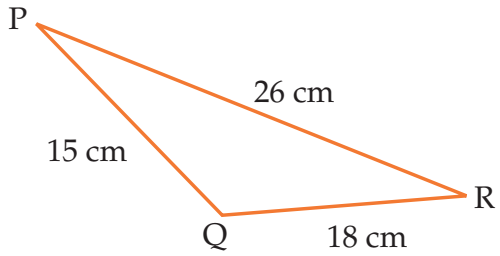
KEY POINTS

1. In any triangle, the shortest side is opposite the smallest angle, and the longest side is opposite the largest angle.
2. If the largest angle is *obtuse* (greater than 90°), then it will have a *negative* cosine.

These two points are demonstrated in the following example.

Example

Calculate the size of the largest angle in this triangle.



Answer

The largest angle in this triangle is $\angle PQR$ (that is, $\angle Q$) because it is opposite the longest side (PR = 26 cm).

We write the cosine rule for this triangle with $\cos(Q)$ as the subject:

$$\cos(Q) = \frac{p^2 + r^2 - q^2}{2pr}$$

$$= \frac{18^2 + 15^2 - 26^2}{2 \times 18 \times 15}$$

$$= \frac{324 + 225 - 676}{540}$$

$$= \frac{^{-}127}{540} = {}^{-}0.2352$$

$$Q = 103.6^\circ \text{ (1 dp)}$$

11.05 Solving triangles

In general, three measurements, including at least one side, are needed to determine the other measurements in a triangle.

Any triangle, obviously, has three sides and three angles. Given three suitable measurements, the process of working out all the other measurements is called **solving the triangle**.

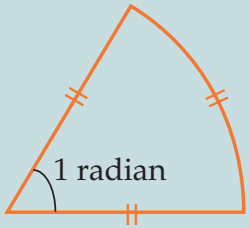
To 'solve' a triangle, use both the sine and cosine rules in a suitable order, as well as the ($\angle$ sum of $\triangle$ is 180°) property.

12.01 Conversion of degrees/radians

Up to now, we have measured angles in **degrees**. This is a rather artificial unit to use. For historical reasons, the convention is that there are 90° in a right-angle and, therefore, 360° in a full turn or complete revolution.

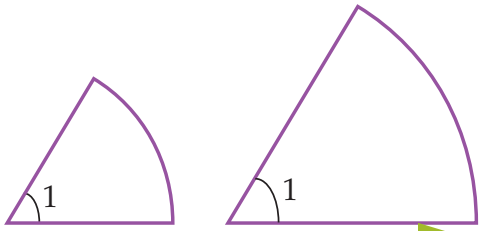
A more natural method is to define a new unit, called a **radian**.

1 radian = the angle formed in a sector with the arc length the same as the radius.



This definition does not depend on the size of the sector. If a sector is drawn with twice the radius and twice the arc length, the angle at the centre is still 1 radian.

A radian, then, is the angle formed when an object has made a circular turn through the same distance as its distance from the centre of the circle.



Radian measure is sometimes called **circular** measure.

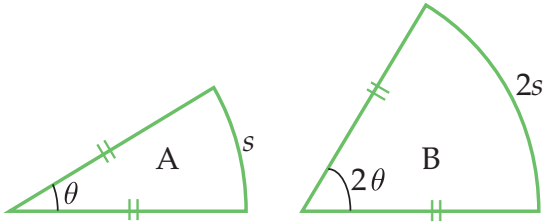
The arc length formula

The length of an arc is proportional to both:

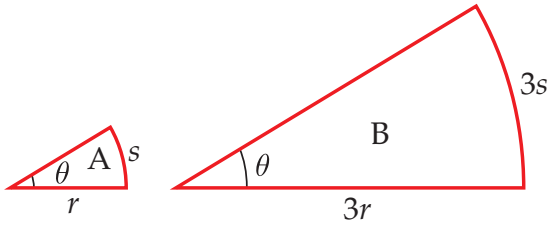
- the angle at the centre of the arc, and
- the radius of the arc.

Consider the diagrams.

1 Two sectors have the same radius.
Sector B has an angle twice the size of sector A. This means the arc length of sector B is twice the arc length of sector A.



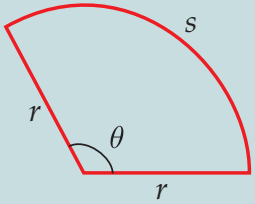
2 Two sectors have the same centre angle.
Sector B has a radius three times the length of sector A. This means the arc length of sector B is three times the arc length of sector A.



Because arc length is proportional to both the angle at the centre of a sector and the radius, we have the **arc length formula**:

$$s = r\theta$$

where s is the arc length, r is the length of the radius and θ is the angle at the centre of the sector, measured in radians.



How do we calculate radians? The answer lies in using the equation for the circumference of a circle.

Conversion of degrees/radians

Rearranging the arc length formula ($s = r\theta$), we get:
$\theta = \frac{s}{r}$

If we take a full turn (360°), the arc length is just the circumference:
$360° = \frac{2\pi r}{r} = 2\pi$

Thus $360° = 2\pi$ (about $2 \times 3.142 = 6.284$ radians) and, in particular, $180° = \pi$ radians.

$$1° = \frac{\pi}{180} \quad \text{and} \quad \pi \text{ radians} = 180°$$

To **convert degrees to radians**, multiply by $\frac{\pi}{180}$.

Often it is best to leave your answer in terms of π.

Example

Convert $45°$ to radians. Leave your answer in terms of π.

Answer

$$45° = 45 \times \frac{\pi}{180} = \frac{45}{180} \times \pi = \frac{1}{4} \times \pi = \frac{\pi}{4}$$

To **convert radians to degrees**, multiply by $\frac{180}{\pi}$.

Example

Convert $\frac{2\pi}{3}$ radians to degrees.

Answer

$$\frac{2\pi}{3} = \frac{2 \times 180°}{3} = \frac{360°}{3} = 120°$$

12.02 Using the arc length formula

Example

Calculate the length of the arc ABC.

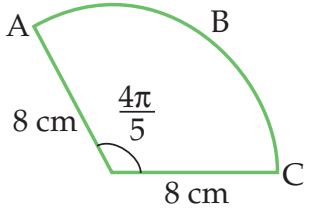
Answer

Use the arc length formula $s = r\theta$.

$$\text{ABC} = r\theta$$

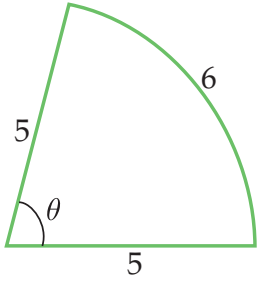
$$= 8 \times \frac{4\pi}{5}$$

$$= \frac{32\pi}{5} \text{ cm} \qquad \text{(or, in decimal form, 20.1 cm)}$$



Example

Calculate the size of the angle labelled θ, in degrees.



Answer

From the arc length formula:

$$\theta = \frac{s}{r} = \frac{6}{5} = 1.2 \text{ radians}$$

$$1.2 \text{ radians} = 1.2 \times \frac{180}{\pi}$$

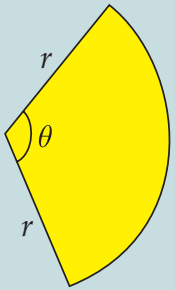
$$= 68.8°$$

12.04 The area of a sector

The area of a sector is given by the formula:

$$\text{Area} = \frac{1}{2} r^2\theta.$$

(Note: θ is in radians.)



Remember that the area of a full circle is given by πr^2. The fraction $\frac{\theta}{2\pi}$ gives the proportion of the sector to the circle. Hence, the area of the sector is given by:

$$\text{Area} = \frac{\theta}{2\pi} \times \pi r^2$$

$$= \frac{\theta r^2}{2} \text{ or } \frac{1}{2} r^2\theta$$

The sector area formula can be rearranged to make either θ or r the subject:

$$A = \frac{1}{2}r^2\theta$$

$$2A = r^2\theta$$

$$\theta = \frac{2A}{r^2} \quad \text{and} \quad r = \sqrt{\frac{2A}{\theta}}$$

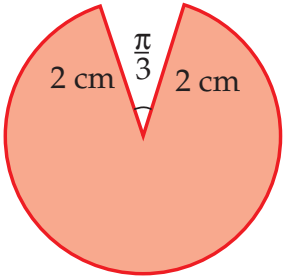
Example

A sector with a centre angle of $\frac{\pi}{3}$ radians has been removed from a circle. The radius is 2 cm. Calculate the area of the remaining (shaded) region.

Answer

The angle at the centre of the shaded sector is:

$$2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$



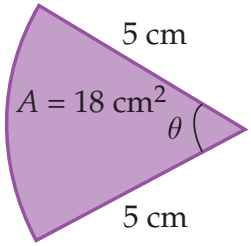
$$\text{Area of sector} = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2} \times 2^2 \times \frac{5\pi}{3}$$

$$= \frac{10\pi}{3} \text{ cm}^2 \quad \text{or} \quad 10.47 \text{ cm}^2 \text{ (4 sf)}$$

Example

Calculate the angle at the centre of this sector. Give the answer in **a** radians and **b** degrees.



Answer

a $\theta = \dfrac{2A}{r^2}$

$= \dfrac{2 \times 18}{5^2}$

$= \dfrac{36}{25}$

$= 1.44$ radians

b To change from radians to degrees, multiply by $\dfrac{180}{\pi}$.

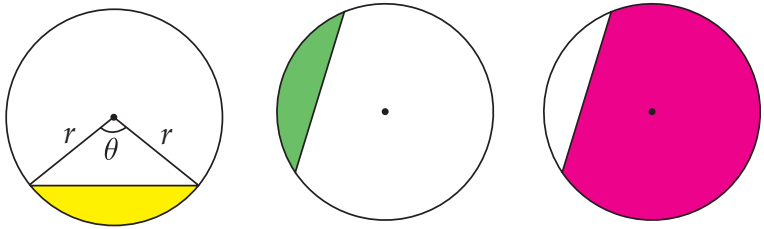
$1.44 \text{ radians} = 1.44 \times \dfrac{180^\circ}{\pi}$

$= 82.5^\circ$

12.06 The area of a segment

A **segment** of a circle is the region between a chord and the circumference. We can define a segment by giving the radius, r, of the circle it is a part of, and also the centre angle, θ.

To calculate the area of a segment, we start with the area of a sector with radius r and centre angle θ, and subtract the area of a triangle from it. The triangle is isosceles, because two sides are a radius of the circle, and the angle between the two sides is θ.

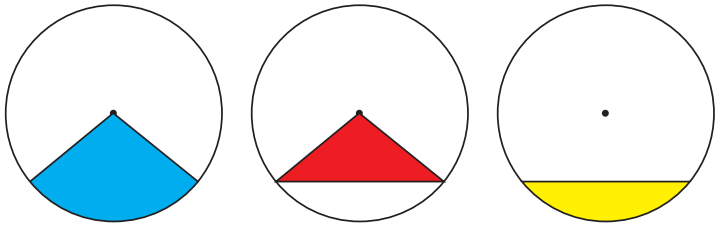


Minor segment

Major segment

Area of sector $= \frac{1}{2} r^2\theta$

Area of triangle $= \frac{1}{2} ab \sin(C) = \frac{1}{2} \times r \times r \times \sin(\theta) = \frac{1}{2} r^2 \sin(\theta)$



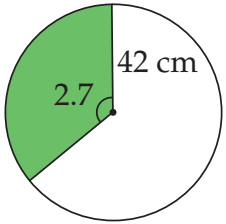
Sector – Triangle = Segment

The area of the segment is given by:

$\frac{1}{2} r^2\theta - \frac{1}{2} r^2 \sin(\theta) = \frac{1}{2} r^2 \left(\theta - \sin(\theta)\right)$

Example

Use the formula $A = \frac{1}{2} r^2 \left(\theta - \sin(\theta)\right)$ to calculate the area of the given segment.



Answer

$r = 42$ cm, $\theta = 2.7$

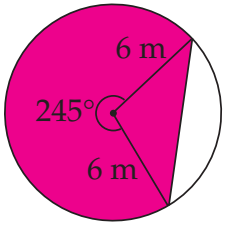
$$\begin{aligned}\text{Area} &= \frac{1}{2} r^2 \left(\theta - \sin(\theta)\right) \\ &= \frac{1}{2} \times 42^2 \times (2.7 - 0.4274) \\ &= 882 \times 2.2726 \\ &= 2004 \text{ cm}^2 \quad (4 \text{ sf})\end{aligned}$$

1 The absence of units for the centre angle indicates radians are being used.

2 When evaluating on a calculator, make sure it is set to radians, usually indicated by 'RAD' on the display.

Example

Calculate the area of the shaded segment.



Answer

First note the angle at the centre is given in degrees. Change it to radians as follows:

$$245° = 245 \times \frac{\pi}{180} = 4.276 \text{ radians}$$

$$\begin{aligned}\text{Area} &= \frac{1}{2} r^2 \left(\theta - \sin(\theta)\right) \\ &= \frac{1}{2} \times 6^2 \times (4.276 - {}^{-}0.9063) \\ &= 18 \times 5.1824 \\ &= 93.28 \text{ m}^2 \quad (4 \text{ sf})\end{aligned}$$

The formula works correctly when the centre angle is reflex. You could check that this answer is the same as if you approached the problem by subtracting the area of the minor segment with centre angle 115° from the area of the whole circle.

13.01 Expansions

When an expression in brackets has a number or pronumeral in front, the brackets can be **expanded** by multiplying the terms inside the brackets, in turn, by the number or pronumeral at the front. This process removes the brackets.

When writing formulae in algebra, we use letters of the alphabet (x, y, p, etc.) to represent numbers that can be substituted in the formula or rule. Because the letter, usually x, is 'standing in' for a number, we call it a **pronumeral.**

Example

Expand $6(x + 2y)$.

Answer

$6(x + 2y) = 6 \times x + 6 \times 2y = 6x + 12y$

Example

Expand $x(3 - x)$.

Answer

$x(3 - x) = 3x - x^2$

Where some terms have the same pronumeral or pronumeral combination, these **like terms** can be 'simplified' by adding or subtracting.

Example

Expand and then simplify
$2(x - 3) + 3(5x + 1)$.

Answer

$2(x - 3) + 3(5x + 1)$
$= 2x - 6 + 15x + 3$
$= 17x - 3$

Take particular care with minus signs in front of brackets. A minus sign changes the sign of **all** the terms inside the brackets. By itself in front of brackets, a minus sign means multiply by $^{-}1$.

Example

Expand and simplify
$2(a - 3b + c) - (a + 6b - c)$.

Answer

$2(a - 3b + c) - (a + 6b - c)$
$= 2a - 6b + 2c - a - 6b + c$
$= a - 12b + 3c$

13.02 Quadratic expansions

A **quadratic** expression in x is one that can be expressed as $ax^2 + bx + c$, where $a \neq 0$.

In a quadratic expression, the highest power of x is x^2. An example of a quadratic expression is $x^2 - 5x + 7$. When two linear expressions in x (of the form $px + q$) are multiplied together, a quadratic is the result.

Example

Expand $(x + 4)(x + 3)$.

Answer

Consider the $(x + 4)$ to be a single term (temporarily).

Then $(x + 4)(x + 3)$

$= (x + 4)x + (x + 4)3$

$= x^2 + 4x + 3x + 12$

$= x^2 + 7x + 12$

Note: with practice, the first step can be left out.

Example

Expand $(x - 1)(x + 8)$.

Answer

$$(x - 1)(x + 8) = x^2 - x + 8x - 8$$
$$= x^2 + 7x - 8$$

Example

Expand $(6 - x)(2x + 3)$.

Answer

$$(6 - x)(2x + 3) = 12x - 2x^2 + 18 - 3x$$
$$= {}^{-}2x^2 + 9x + 18$$

When expanding quadratics, the convention is to write the resulting expression with 'descending' powers of x, i.e. start with the x^2 term.

13.03 Perfect squares

When both brackets in a quadratic expression are the same, we say that the expression can be written in **perfect square form**. Instead of repeating the same bracket more than once, we write the expression as a power.

A perfect square can be written in the form $(ax + b)^2$.

Examples

$(x - 3)(x - 3)$ is better written as $(x - 3)^2$.

$(4x + 5)(4x + 5)$ is written as $(4x + 5)^2$.

Perfect squares can be expanded in exactly the same way as other quadratics.

Example

Expand

a $(x-3)^2$

b $(4x+5)^2$.

Answer

a $(x-3)^2 = (x-3)(x-3)$

$= x^2 - 3x - 3x + 9$

$= x^2 - 6x + 9$

b $(4x+5)^2 = (4x+5)(4x+5)$

$= 16x^2 + 20x + 20x + 25$

$= 16x^2 + 40x + 25$

13.04 Difference of two squares

When the expressions in both pairs of brackets in a quadratic expression are the same except for different signs, the resulting expression is called a **difference of two squares**. In general:

- $(x + y)(x - y)$ expands to $x^2 - y^2$
- $(px + q)(px - q)$ expands to $p^2x^2 - q^2$
- $(ax + by)(ax - by)$ expands to $a^2x^2 - b^2y^2$.

Example

Expand

a $(x + 2)(x - 2)$ **b** $(4x - 3)(4x + 3)$.

Answer

a $(x + 2)(x - 2) = x^2 + 2x - 2x - 4$

$= x^2 - 4$

b $(4x - 3)(4x + 3) = 16x^2 - 12x + 12x - 9$

$= 16x^2 - 9$

13.05 Simplifying like terms after expansion

Example

Expand and simplify
$x(5x + 4) - (2x + 1)(2x - 1)$.

Answer

$x(5x + 4) - (2x + 1)(2x - 1)$
$= 5x^2 + 4x - (4x^2 - 1)$
$= 5x^2 + 4x - 4x^2 + 1$
$= x^2 + 4x + 1$

13.06 Expanding more than two brackets

Example

Expand and simplify
$(x + 1)(x - 2)(x + 4)$.

Answer

Keep one bracket unchanged and expand the other two:

$(x + 1)(x - 2)(x + 4) = (x + 1)(x^2 + 2x - 8)$

Now multiply the terms in the second bracket in turn by both terms in the first bracket:

$$= x(x^2 + 2x - 8) + 1(x^2 + 2x - 8)$$
$$= x^3 + 2x^2 - 8x + x^2 + 2x - 8$$
$$= x^3 + 3x^2 - 6x - 8$$

Example

Expand and simplify $(x - 5)^3$.

Answer

$$
\begin{aligned}
(x - 5)^3 &= (x - 5)(x - 5)(x - 5) \\
&= (x - 5)(x^2 - 10x + 25) \\
&= x^3 - 10x^2 + 25x - 5x^2 + 50x - 125 \\
&= x^3 - 15x^2 + 75x - 125
\end{aligned}
$$

13.07 & 13.08 Linear equations

A **linear equation** can be reduced eventually to the form $ax = b$. Linear equations generally have only one solution. (There are exceptions to this rule – see later.)

KEY POINTS

To solve a linear equation, aim to get pronumerals (x) to the left-hand side (LHS) of the equals (=) sign and numbers to the right-hand side (RHS). Then try to get x by itself. Here are the steps to follow.

1. Expand any brackets.
2. Simplify, if necessary, by adding or subtracting like terms.
3. Transfer terms with x to the LHS and number-only terms to the RHS. When you do this, you have to change their sign: + changes to –, and – changes to +. '*Change sides, change operations.*'
4. Simplify, if necessary, by adding or subtracting like terms.
5. Divide by the number in front of x.

Example

Solve the equation $3(x - 1) = 5x + 17$.

Answer

$$
\begin{aligned}
3(x-1) &= 5x + 17 \\
3x - 3 &= 5x + 17 && \text{(expanding brackets)} \\
3x - 5x &= 17 + 3 && \text{(The } 5x \text{ is moved to the LHS and has its sign changed to become } {}^{-}5x\text{. Likewise, the } -3 \text{ becomes } +3 \text{ when moved to the other side.)} \\
{}^{-}2x &= 20 && \text{(simplifying)} \\
x &= {}^{-}10 && \text{(dividing by } {}^{-}2\text{, the number before } x\text{)}
\end{aligned}
$$

KEY POINTS

Not every linear equation has a unique solution

Some equations have an infinite number of solutions. Any number will make the equation true.

Example

Solve the equation

$6x + 5 = 2(3x + 5) - 5$.

Answer

$$6x + 5 = 2(3x + 5) - 5$$
$$6x + 5 = 6x + 10 - 5$$
$$6x + 5 = 6x + 5$$
$$6x - 6x = 5 - 5$$
$$0 = 0$$

which is (trivially) always true.

It follows that any number could be substituted for x.

We say the **solution set** is $\mathbb{R}$, the real numbers.

Other equations have no solution at all. We say the solution set is the **empty set** – i.e. { } or Ø.

Example

Solve the equation $2x - 3 = 2x + 7$.

Answer

$$2x - 3 = 2x + 7$$
$$2x - 2x = 7 + 3$$
$$0 = 10 \qquad \text{which is impossible.}$$

The solution set is { }.

Note: it should be obvious that this equation has no solution – doubling a number and subtracting 3 will never give the same result as doubling the number and adding 7.

13.09 Equations in fraction form – 1

If the equation has fractions in it, multiply both sides by the common denominator.

Example

Solve the equation $\frac{x}{4} + 3 = \frac{2x}{7}$.

Answer

The common denominator is 28 (i.e. 4×7). Multiply all three terms by 28.

$$28 \times \frac{x}{4} + 28 \times 3 = 28 \times \frac{2x}{7}$$

$$7x + 84 = 8x$$

$$7x - 8x = {}^{-}84$$

$${}^{-}x = {}^{-}84$$

$$x = 84$$

13.10 Equations in fraction form – 2

When an equation has fractions only on both sides, a useful rule of thumb is to **cross-multiply** – i.e. multiply diagonally.

Example

Solve the equation $\frac{x-4}{5} = \frac{2x}{3}$.

Answer

Cross-multiply:

$$3(x-4) = 10x$$
$$3x - 12 = 10x$$
$$3x - 10x = 12$$
$$^{-}7x = 12$$
$$x = \frac{12}{^{-}7} = \frac{^{-}12}{7}$$

13.11 Inequations

An inequation is like an equation but, instead of having an equals sign (=), it has one of these inequalities:

- $<$ less than
- $>$ greater than
- $\leq$ less than or equal to
- $\geq$ greater than or equal to.

To solve inequations, we use similar methods to those covered earlier. However, often the answer is a set of numbers rather than one single number.

One important rule when solving inequations is that when you divide by a *negative* number at the final step, you **reverse** the inequality. That is, a $<$ becomes a $>$, and a $>$ becomes a $<$ sign.

Example

Solve $4x - 5 < 2$.

Answer

$$4x - 5 < 2$$

$$4x < 2 + 5$$

$$4x < 7$$

$$x < \frac{7}{4} \quad (\text{or } x < 1.75)$$

Example

Solve $4x - 3(x + 6) \geq 5x - 22$.

Answer

$$4x - 3x - 18 \geq 5x - 22$$
$$x - 18 \geq 5x - 22$$
$$x - 5x \geq {}^{-}22 + 18$$
$${}^{-}4x \geq {}^{-}4 \qquad \text{(on division by } {}^{-}4\text{, we}$$
$$x \leq 1 \qquad \text{change the inequality sign)}$$

13.13 Basic indices

Repeated multiplication of the same number can be shown by using **power form**. For example:
$2 \times 2 \times 2 \times 2 \times 2 = 2^5$.

In general, in an expression such as b^p, b is called the **base** and p is called the **index** or **exponent**.

Powers are evaluated either by repeated multiplication, or by using the special 'power' key on calculators – i.e. x^y , or y^x on some models.

Examples

$4.1^3 = 68.921$

$(^-3)^5 = {}^-243$

When the base is *negative:*

- an *odd* number of negative numbers multiply to a negative product
- an *even* number of negative numbers multiply to a *positive* product.

That is, if the index is odd, the answer is negative; if the index is even, the answer is positive.

Example

Evaluate the following.

a $(^{-}9)^4$ **b** $(^{-}2)^5$

Answer

a $(^{-}9)^4 = 9^4 = 6561$ **b** $(^{-}2)^5 = {}^{-}2^5 = {}^{-}32$

13.14–13.16 Power laws

Multiplication

When *multiplying* numbers with the same base in power form, *add* the indices.

Example

$4^3 \times 4^5 = 4^8$

$x \times x^6 = x^7$

If there are numbers in front of the bases, they are multiplied.

Example

$2x^5 \times 10x^6 = 20x^{11}$

Division

When *dividing* numbers with the same base in power form, *subtract* the indices.

Examples

$$\frac{x^6}{x^2} = x^4$$

$$\frac{x^3}{x^4} = \frac{1}{x}$$

If there are numbers in front of the base, they are divided or simplified.

Examples

$$\frac{24x^{14}}{3x^2} = 8x^{12}$$

$$\frac{28x}{35x^3} = \frac{4}{5x^2}$$

Powers of powers

When a base to a given index is *raised to another index*, we simplify by *multiplying* the indices.

Example

$(x^5)^2 = x^{10}$

If there are numbers in the brackets, they are raised to the given power.

Example

$(3x^6)^5 = 243x^{30}$

14.01 Simplifying

Common factors should always be taken out to simplify a fraction.

Example

Simplify $\frac{2cd}{4de}$.

Answer

This fraction has common factors of 2 and d.

'Cancelling' each common factor gives:

$$\frac{2cd}{4de} = \frac{{}^{1}2cd}{{}^{2}4de}$$

$$= \frac{c}{2e}$$

14.02 Multiplication

To multiply two fractions, multiply the numerators and denominators separately.

Example

$$\frac{4b}{5c} \times \frac{3c}{2d} = \frac{12bc}{10cd} = \frac{6b}{5d}$$

14.03 Division

To divide two fractions, multiply the first fraction by the reciprocal of the second fraction. Note: to form the reciprocal of a fraction, turn the fraction upside-down.

Example

$$\frac{2x}{y} \div \frac{5x}{3y} = \frac{2x}{y} \times \frac{3y}{5x} = \frac{6xy}{5xy} = \frac{6}{5}$$

14.05 & 14.06 Addition and subtraction of algebraic fractions

The simplest cases of addition and subtraction involve fractions with the *same* denominator. In the same way as with ordinary fractions, just add the numerators.

Example

Add $\frac{5x}{p} + \frac{3x}{p}$.

Answer

$\frac{8x}{p}$

Fractions can only be added/subtracted when the denominators are the same. When the denominators are *different*, obtain the common denominator by multiplying. Then the numerators have to be adjusted by 'cross-multiplying' diagonally.

Example

Simplify (write as one fraction) $\frac{x}{3} + \frac{2x}{p}$.

Answer

The common denominator is $(3 \times p) = 3p$. Cross-multiply to obtain respective numerators px (i.e. $p \times x$) and $6x$ (i.e. $3 \times 2x$).

$$\frac{x}{3} + \frac{2x}{p} = \frac{px}{3p} + \frac{6x}{3p}$$

$$= \frac{px + 6x}{3p}$$

Harder examples involve denominators that already have a common factor, so the common denominator is not necessarily the product of the denominators.

Example

Simplify $\frac{2x+1}{6x} + \frac{5}{4x}$.

Answer

Here the most suitable common denominator is $12x$. To obtain this common denominator, multiply the first fraction top and bottom by 2, and the second fraction top and bottom by 3.

$$\frac{2x+1}{6x} + \frac{5}{4x} = \frac{4x+2}{12x} + \frac{15}{12x}$$

$$= \frac{4x+2+15}{12x}$$

$$= \frac{4x+17}{12x}$$

Other examples have linear factors that, when multiplied, give a *quadratic expression* for the common denominator.

Example

Simplify $\frac{4}{x-3} - \frac{3}{x+1}$.

Answer

The common denominator is
$(x-3)(x+1) = x^2 - 2x - 3$.

4 times $(x+1)$ gives $4x+4$ as the first numerator and 3 times $(x-3)$ gives $3x-9$ as the second numerator.

$$\frac{4}{x-3} - \frac{3}{x+1} = \frac{4x+4-(3x-9)}{x^2-2x-3}$$

$$= \frac{4x+4-3x+9}{x^2-2x-3}$$

$$= \frac{x+13}{x^2-2x-3}$$

15.01 Factorising

The reverse process to expanding brackets is where a sum is changed to a product involving brackets. This is called **factorising**.

Example

Factorise the following.

a $5a - 10b$

b $x^2 - 2x$

c $3x^2 + 9x$

d $3x + 6y + 12$

Answer

a $5a - 10b = 5(a - 2b)$ (Common factor is 5.)

b $x^2 - 2x = x(x - 2)$ (Common factor is x.)

c $3x^2 + 9x = 3x(x + 3)$ (There are *two* common factors – both must be taken outside the brackets.)

d $3x + 6y + 12 = 3(x + 2y + 4)$ (Common factor is 3.)

When factorising, always look for a common factor first.

15.02 Two-stage factorising

In some types of problem, factorising needs to be done in two stages.

Example

Factorise $pr + qr + ps + qs$.

Answer

Clearly, there is no single factor common to all terms. So, factorise in pairs.

$pr + qr + ps + qs = r(p + q) + s(p + q)$

Now note that $r(p + q) + s(p + q)$ has a common factor of $p + q$:

$= (p + q)(r + s)$

Instead of looking for a common factor for all terms, examine pairs for a common factor.

Example

Factorise $wx + zy - wy - xz$.

Answer

Rewrite as $wx - wy + zy - xz$

$= w(x - y) + z(y - x)$

$= w(x - y) - z(x - y)$ Note: $(y - x) = {}^{-}(x - y)$

$= (w - z)(x - y)$

Some rearrangement may be needed first, to obtain convenient pairs.

15.03 Factorising quadratics

Recall the expansion:

$(x + 3)(x - 2) = x^2 + x - 6$.

Another expansion is:

$(2x - 3)^2 = 4x^2 - 12x + 9$.

Expansions like these, with one variable, x, and two pairs of brackets each with linear expressions, are often called **quadratic expansions**. The result of the expansion is an expression that has x^2 as the highest power of x.

The numbers in front of the powers of x are called **coefficients**. The number by itself, usually at the end, is called the **constant term**.

Example

In the quadratic $x^2 + 3x - 2$:

- the coefficient of x^2 is 1
- the coefficient of x is 3
- the constant term is $^-2$.

Quadratics have two linear factors. If the coefficient of x^2 is 1, then each of the factors in brackets will start with x, i.e. $(x\quad)(x\quad)$.

The key to factorising quadratics is to work out the numbers at the end of each pair of brackets.

Example

Factorise $x^2 + 8x + 12$.

Answer

Write some pairs of numbers that multiply to 12:

1, 12 2, 6 3, 4

The pair that adds to 8 is 2 and 6, so the factorisation is $(x + 2)(x + 6)$.

Note: the order of the two factors is not important – the answer could also be $(x + 6)(x + 2)$.

The rule is that the numbers we are looking for:

- multiply to the constant term (the number at the end), and
- add to the coefficient of x (the number in the middle).

If the coefficient of x and the constant term are not both positive, more care needs to be taken.

Example

Factorise $x^2 - 3x - 10$.

Answer

Possible pairs of numbers are:

$10, {}^{-}1 \quad 5, {}^{-}2 \quad 2, {}^{-}5 \quad 1, {}^{-}10$

The pair that adds to $^{-}3$ is 2 and $^{-}5$, so the two factors are $(x + 2)(x - 5)$.

Example

Factorise $x^2 - 6x + 5$.

Answer

Possible pairs of numbers are:

5, 1 and $^{-}5, ^{-}1$.

The pair that adds to $^{-}6$ is $^{-}1$ and $^{-}5$, so the two factors are $(x - 1)(x - 5)$.

15.04 Quadratics where the coefficient of x^2 is not 1

It is more difficult to factorise quadratics where the coefficient of x^2 is not 1. This is because the rule about the two numbers at the end of each bracket adding to the coefficient of x no longer works.

One approach is to look for a common factor first, and then factorise.

Example

Factorise $2x^2 - 10x - 12$.

Answer

On inspecting this quadratic, there is a common factor of 2.

$$\begin{aligned} 2x^2 - 10x - 12 &= 2(x^2 - 5x - 6) \\ &= 2(x - 6)(x + 1) \end{aligned}$$

15.05 Harder quadratic factorising

If there is no common factor in a quadratic expression, 'guess and check' methods can be used.

Recall that the x terms inside each pair of brackets have to multiply to give the x^2 term in the expansion. The numbers inside each pair of brackets have to multiply to give the constant term of the expansion.

One approach is to examine pairs of brackets that fit these conditions. Expand each pair to discover which one gives the expansion you are trying to factorise. (It is always a good idea to expand the brackets to check your answer.)

Example

Factorise $4x^2 + 11x + 6$.

Answer

Possible pairs of factors that give $4x^2$ and 6 are:

$(4x + 6)(x + 1)$ $\quad$ $(4x + 1)(x + 6)$

$(4x + 3)(x + 2)$ $\quad$ $(4x + 2)(x + 3)$

$(2x + 6)(2x + 1)$ $\quad$ $(2x + 1)(2x + 6)$

$(2x + 2)(2x + 3)$ $\quad$ $(2x + 3)(2x + 2)$

The only pair that gives $4x^2 + 11x + 6$ when expanded is $(4x + 3)(x + 2)$.

With some experience, a feeling for the numbers involved gradually develops. You will soon find that it is not necessary to write out all possible pairs of factors.

KEY POINTS

Here's a two-stage method for working out the factors of quadratics where the coefficient of x^2 is not 1.

1. Recall that the quadratic can be written as $ax^2 + bx + c$.
2. Look for two numbers that:
   - multiply to ac, and
   - add to b.
3. Call these numbers p and q.
4. Write $ax^2 + bx + c$ as $ax^2 + px + qx + c$.
5. Now factorise $ax^2 + px + qx + c$ in two stages.

Example

Factorise $12x^2 + x - 6$.

Answer

Two numbers that multiply to $^{-}72$ $(= 12 \times {}^{-}6)$
and add to 1 are 9 and $^{-}8$.

Write $12x^2 + x - 6$ as $12x^2 + 9x - 8x - 6$.

$$\begin{aligned} 12x^2 + x - 6 &= 12x^2 + 9x - 8x - 6 \\ &= 3x(4x + 3) - 2(4x + 3) \\ &= (3x - 2)(4x + 3) \end{aligned}$$

(*Source*: This method came from Ross Porter of Cambridge, New Zealand.)

15.06 Two special cases of quadratic factorisation

1 Perfect squares

When both pairs of brackets are the same, we get an expression called a **perfect square**.

Example

Factorise $x^2 - 6x + 9$.

Answer

$$x^2 - 6x + 9 = (x - 3)(x - 3)$$
$$= (x - 3)^2$$

Example

Factorise $x^2 + 16x + 64$.

Answer

$x^2 + 16x + 64 = (x + 8)(x + 8) = (x + 8)^2$

2 Difference of two squares

An expression such as $a^2 - b^2$ is called the **difference of two squares**. In this type of expression, there is no term in x.

The factorisation always follows the same pattern. Both factors are identical except for the signs, one of which is $+$ and the other $-$.

Example

Factorise $x^2 - 49$.

Answer

We want two numbers that multiply to $^-49$ and add to zero.

Thus, $x^2 - 49 = (x - 7)(x + 7)$.

15.07 Using common factors

Some fractions can be simplified by looking for common factors and then cancelling.

Example

Simplify $\dfrac{x^2 + x - 2}{x - 1}$.

Answer

$$\frac{x^2 + x - 2}{x - 1} = \frac{(x - 1)(x + 2)}{(x - 1)} = x + 2$$

Be aware of disguised changes of sign. For example, in the following fraction, the factor $(x - 7)$ cancels with the factor $(7 - x)$ leaving $^{-}1$ in the answer.

$$\frac{x - 7}{7 - x} = \frac{^{-}1(7 - x)}{7 - x} = {}^{-}1$$

Example

Simplify $\frac{16 - x^2}{x + 4}$.

Answer

$$\frac{16 - x^2}{x + 4} = \frac{(4 + x)(4 - x)}{x + 4} = 4 - x$$

Note: $4 + x = x + 4$, so these are common factors and cancel.

Example

Simplify $\dfrac{x^2 - 5x + 6}{8 - 2x - x^2}$.

Answer

$$\frac{x^2 - 5x + 6}{8 - 2x - x^2} = \frac{(x - 3)(x - 2)}{(2 - x)(x + 4)}$$

$$= \frac{x - 3}{x + 4} \times {}^{-}1$$

$$= \frac{3 - x}{x + 4}$$

16.01 Linear rearrangements

Often, a formula has to be rearranged to make one of the other variables the subject. The same rules apply as in solving equations. Whatever is done to one side of the formula should be done to the whole of the other side.

Example

The formula for changing degrees Fahrenheit to degrees Celsius is $C = \frac{5}{9}(F - 32)$.

Make F the subject of this formula.

Answer

$$C = \frac{5}{9}(F - 32)$$
$$9C = 5(F - 32)$$
$$9C = 5F - 160$$
$$5F = 9C + 160$$
$$F = \frac{9C + 160}{5}$$

Notice how *all* of $9C + 160$ is divided by 5. The fraction line means we don't need to use brackets, but they are implied.

That is: $\frac{9C + 160}{5}$ is the same as $\frac{(9C + 160)}{5}$.

16.02 Applications of linear rearrangements

Example

Some pharmacists use a formula known as Cowling's rule to calculate a child dosage for medicines when the adult dosage is known.

The rule is $d = \frac{a(y+1)}{24}$, where d is the child dosage, a is the adult dosage and y is the age of the child in years.

a Rearrange the equation to make a the subject of Cowling's rule.

b Above what age is it not appropriate to use Cowling's rule? Explain.

Answer

a Clear the fraction by multiplying by 24:

$24d = a(y + 1)$

Then divide by $(y + 1)$ to get a by itself:

$a = \frac{24d}{y+1}$

b The rule is no longer necessary for ages where Cowling's rule gives the same dosage for a child as for an adult (and for older ages too!).

a will equal d when $\frac{24}{y+1} = 1$.

This occurs when $y = 23$.

Note that, in practice, the rule is probably more frequently used for children under 16.

16.03 Rearrangements of rational expressions

A **rational expression** is a fraction where both the numerator and denominator are polynomials.

Examples

$$\frac{2x-3}{x+1} \qquad \frac{6}{x^2-4}$$

All of the problems up until now have had x, the required subject, in one place only. It has been straightforward to isolate x on its own and make it the new subject. However, many relations have x in several places, so new techniques are needed to make x the subject.

When the relation is rational, the best procedure is to:

1. 'cross-multiply' to remove fractions
2. move all x terms to one side, and non-x terms to the other side of the equation
3. factorise, so that x appears only once on the left-hand side of the equation
4. divide, to make x the subject.

Example

Make x the subject of the relation $5y = \dfrac{2}{x-3}$.

Answer

$$5y = \frac{2}{x-3}$$

$$5y(x-3) = 2 \quad \text{(cross-multiplying)}$$

$$5xy - 15y = 2 \quad \text{(rearranging)}$$

$$5xy = 15y + 2$$

$$x = \frac{15y+2}{5y}$$

Example

Make x the subject of the relation $y = \dfrac{3x-1}{x+2}$.

Answer

$$y = \frac{3x-1}{x+2}$$
$$xy + 2y = 3x - 1$$
$$xy - 3x = {}^{-}2y - 1$$
$$x(y-3) = {}^{-}2y - 1$$
$$x = \frac{{}^{-}2y - 1}{y-3}$$

And now a trick to simplify this answer further! The convention is to have as few '–' signs as possible.

Multiply by $\dfrac{{}^{-}1}{{}^{-}1}$: $\quad x = \dfrac{2y+1}{3-y}$

16.04 Rearrangements of formulae involving square roots

To make x the subject of a formula when the x term is inside a square root symbol, *square both sides* to 'undo' the effect of the square root. However, before doing this, check carefully that the *square root is on its own on one side of the equation*.
Consider these two examples.

1 $y = \sqrt{x-1}$

Squaring gives $y^2 = x - 1$.

But:

2 $y = \sqrt{x+5} + 1$

This is similar to $y = A + 1$, which squares to $y^2 = A^2 + 2A + 1$.

Squaring gives:

$$\begin{aligned} y^2 &= \left(\sqrt{x+5}\right)^2 + 2\sqrt{x+5} + 1 \\ &= x + 5 + 2\sqrt{x+5} + 1 \\ &= x + 6 + 2\sqrt{x+5} \end{aligned}$$

which does not remove the square root symbol.

Example

Make x the subject of the formula $y = \sqrt{2x + 5}$.

Answer

$$y = \sqrt{2x + 5}$$

$$y^2 = 2x + 5 \quad \text{(squaring)}$$

$$2x = y^2 - 5$$

$$x = \frac{y^2 - 5}{2}$$

Example

Make x the subject of the formula $4y = \sqrt{3x-1} + 5$.

Answer

$$4y = \sqrt{3x-1} + 5$$

$$4y - 5 = \sqrt{3x-1}$$ (rearranging to isolate the square root term on one side)

$$(4y-5)^2 = 3x - 1$$

$$16y^2 - 40y + 25 = 3x - 1$$

$$3x = 16y^2 - 40y + 26$$

$$x = \frac{16y^2 - 40y + 26}{3}$$

16.05 Rearrangements of formulae involving perfect squares

The usual way to 'undo' the effect of squaring is to take the square root. We can use this approach if there is a single, perfect square term in x in the expression to be rearranged.

Examples

$y = x^2$ $\quad$ $y + 1 = (x - 3)^2$ $\quad$ $5y = (2x + 7)^2$

All of the above expressions include perfect squares in x.

Example

Make x the subject of the formula

$y + 1 = (x - 3)^2$.

Answer

$$y + 1 = (x - 3)^2$$

$$\pm\sqrt{y + 1} = x - 3 \qquad \text{(taking the square root of both sides)}$$

$$x = \pm\sqrt{y + 1} + 3$$

16.06 Further applications of rearrangement

Example

The area of a circle can be calculated from the formula $A = \frac{\pi d^2}{4}$ where d is the length of the diameter.

a Rearrange this formula to make d the subject.

b Calculate the diameter of the circle that has an area of 60 m^2.

Answer

a

$$A = \frac{\pi d^2}{4}$$

$$4A = \pi d^2$$

$$d^2 = \frac{4A}{\pi}$$

$$d = \sqrt{\frac{4A}{\pi}}$$

b

$$d = \sqrt{\frac{4A}{\pi}}$$

$$= \sqrt{\frac{4 \times 60}{\pi}}$$

$$= \sqrt{\frac{240}{\pi}} = \sqrt{76.39}$$

$$= 8.74 \text{ m}$$

17.01–17.03 Solving quadratic equations

Quadratic equations *have two solutions*. It is best to first check whether a quadratic equation can be solved by factorising. Arrange the terms in the equation so that the quadratic expression equals zero.

Example

Solve the equation $x^2 - 5x - 14 = 0$.

Answer

Factorise first to get $(x - 7)(x + 2) = 0$.

There are two possibilities:

either $(x - 7) = 0$ or $(x + 2) = 0$.

If $x - 7 = 0$ then $x = 7$.

If $x + 2 = 0$ then $x = {}^{-}2$.

The solution set is $x = 7$ and $x = {}^{-}2$.

Example

Solve $x^2 + 2x = 24$.

Answer

Rearrange the equation so that all terms are on the left-hand side, with 0 on the right-hand side.

$x^2 + 2x - 24 = 0$

$(x + 6)(x - 4) = 0$

$x = {}^{-}6$ or $x = 4$

It is also possible to reconstruct the quadratic equation if we know the solutions.

Example

A quadratic equation has solutions $x = {}^{-}4$ and $x = \frac{1}{2}$. Express the equation in the form $ax^2 + bx + c = 0$.

Answer

If $x = {}^{-}4$ is a solution, then $x + 4 = 0$, so one of the factors is $(x + 4)$.

If $x = \frac{1}{2}$ is a solution, then $x - \frac{1}{2} = 0$, and we can simplify and remove the fraction by multiplying by 2. This gives $2x - 1 = 0$.

The full equation can be written in factorised form as $(x + 4)(2x - 1) = 0$.

Expanding gives $2x^2 - x + 8x - 8 = 0$, or $2x^2 + 7x - 8 = 0$.

There is an infinite number of quadratic equations with these two solutions. The best answer is one where a, b and c are integers and have no common factor.

17.04 Expressions that include quadratics

Sometimes, it is possible to factorise polynomial expressions that contain quadratics further – that is, to write the original expression as a product of linear factors only.

Each linear factor of a polynomial equation gives one solution – so, for example, an equation with three linear factors will have three solutions, or roots.

Example

Solve the equation

$(x + 2)(x^2 - 4x - 5) = 0$.

Answer

$(x + 2)(x^2 - 4x - 5) = 0$

$(x + 2)(x - 5)(x + 1) = 0$

$x = -2$ or $x = 5$ or $x = -1$

Sometimes, the solutions of a quadratic equation are called **roots**. At this level, we use the terms 'solutions' and 'roots' interchangeably. The term 'root' is used in advanced mathematics; you will see roots mentioned frequently if you take Level 3 algebra and study complex numbers.

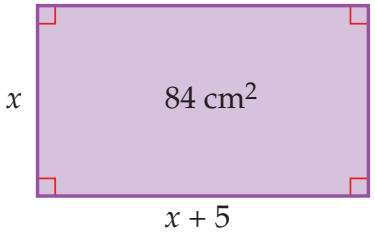
17.05 Applications of quadratic equations

Many problems in mathematics lead naturally to the solution of quadratic equations. Several types of problem come from geometry and area.

Example

One side of a rectangle is 5 cm longer than the other. The area is 84 cm^2. Calculate the lengths of the sides.

Answer



Let the shorter side be x. Then the longer side is $(x + 5)$.

Area = short side × long side

$$x(x + 5) = 84$$
$$x^2 + 5x = 84$$
$$x^2 + 5x - 84 = 0$$
$$(x + 12)(x - 7) = 0$$
$$x = {}^{-}12 \text{ or } 7$$

Reject the answer of $^{-}12$ as it is not a possible length.

That is, the shorter side is $x = 7$ cm, and the longer side is $(x + 5) = 7 + 5 = 12$ cm.

Example

The squares of two consecutive whole numbers add to 145.
What are the two numbers?

Answer

Consecutive numbers have a difference of 1.

Two consecutive numbers can be written as x and $x + 1$.

Given: $x^2 + (x + 1)^2 = 145$

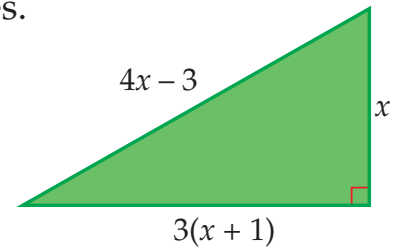
$$x^2 + x^2 + 2x + 1 = 145$$
$$2x^2 + 2x - 144 = 0$$
$$x^2 + x - 72 = 0$$
$$(x + 9)(x - 8) = 0$$
$$x = {}^{-}9 \text{ or } 8$$

Reject $x = {}^{-}9$ as it is not a whole number.

$x = 8$, and the other number is 9.

Example

The right-angled triangle drawn here has sides x, $3(x + 1)$ and $4x - 3$. Find x, and hence write the lengths of the three sides.



Answer

From the Pythagoras relation:

$$(4x - 3)^2 = x^2 + [3(x + 1)]^2$$
$$(4x - 3)^2 = x^2 + (3x + 3)^2$$
$$16x^2 - 24x + 9 = x^2 + 9x^2 + 18x + 9$$
$$6x^2 - 42x = 0$$
$$x^2 - 7x = 0$$
$$x(x - 7) = 0$$
$$x = 0 \text{ or } 7$$

Clearly $x = 0$ is not a possible value.

The three sides are 7, 24 and 25.

17.06 Solving quadratic equations by completing the square

Quadratic equations that do not factorise can sometimes be solved by writing them in the form: $(x + p)^2 = q$ or $(x - p)^2 = q$.

Then, the solutions can be obtained by taking the square root and rearranging.

The process of rearranging a quadratic expression so that it includes a perfect square is called **completing the square**.

The expression $(x + p)^2$ is called a **perfect square**.

KEY POINTS

These steps show the method for solving the equation $x^2 + bx + c = 0$ by completing the square.

- Take the constant (c) and transpose it to the other side of the $=$ sign (remember the sign change).
- Add $\left(\frac{b}{2}\right)^2$ to both sides. This is (half the coefficient of x)^2.
- Write the left-hand side as a perfect square.
- Take the square root of both sides (remember to use both positive and negative values).
- Rearrange.

Example

Solve the equation $x^2 - 8x - 17 = 0$ by completing the square.

Answer

$x^2 - 8x - 17 = 0$

$x^2 - 8x = 17$

$x^2 - 8x + 16 = 17 + 16$ Note: the 16 is

$(x - 4)^2 = 33$ (half of $^-8)^2$.

$x - 4 = \sqrt{33} = \pm 5.745$

$x = 5.745 + 4$ or $^-5.745 + 4$

$x = 9.745$ or $^-1.745$

Generally, the quadratic equation will be of the form $ax^2 + bx + c = 0$.

To solve the quadratic equation $ax^2 + bx + c = 0$ by completing the square, the first step should be to divide through by a, so that the equation has a coefficient of x^2 equal to 1.

The other steps are the same as for solving $x^2 + bx + c = 0$.

Example

Solve the equation $5x^2 + 8x + 1 = 0$ by completing the square.

Answer

$$5x^2 + 8x + 1 = 0$$

$$x^2 + \frac{8}{5}x + \frac{1}{5} = 0$$

$$x^2 + \frac{8}{5}x = \frac{-1}{5}$$

$$x^2 + \frac{8}{5}x + \left(\frac{4}{5}\right)^2 = \frac{-1}{5} + \left(\frac{4}{5}\right)^2$$

$$\left(x + \frac{4}{5}\right)^2 = \frac{-5 + 16}{25}$$

$$\left(x + \frac{4}{5}\right)^2 = \frac{11}{25}$$

$$x + \frac{4}{5} = \sqrt{\frac{11}{25}} = \pm 0.6633$$

$$x = 0.6633 - 0.8 \quad \text{or} \quad -0.6633 - 0.8$$

$$x = -0.1367 \quad \text{or} \quad -1.4633$$

17.07 The quadratic formula

There is a special formula for solving quadratic equations. We use this formula when a quadratic equation does not factorise easily.

In general, if the **quadratic equation** is $ax^2 + bx + c = 0$, then the two solutions are:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We can summarise these two results in a single formula by using the symbol $\pm$. This symbol means either $+$ or $-$ in the same position.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This symbol saves us writing the formula twice, firstly with a $+$ sign, and again with a $-$ sign.

To find one solution, use the formula with the $+$ sign; then to find the other solution, use the same formula but with the $-$ sign.

Example

Use the quadratic formula to solve the quadratic equation $2x^2 - 5x + 1 = 0$.

Answer

Here $a = 2$, $b = {}^{-}5$, $c = 1$.

$$x = \frac{{}^{-}b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{5 \pm \sqrt{({}^{-}5)^2 - 4 \times 2 \times 1}}{2 \times 2}$$

$$= \frac{5 \pm \sqrt{25 - 8}}{4}$$

$$= \frac{5 \pm \sqrt{17}}{4}$$

Now take each case:

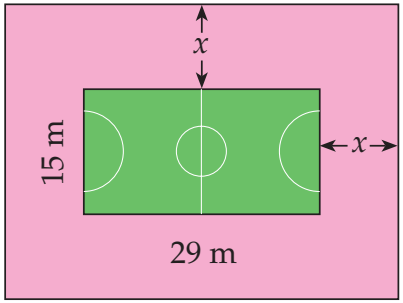
$$\frac{5 + \sqrt{17}}{4} = 2.281 \text{ (3 dp)}$$

$$\frac{5 - \sqrt{17}}{4} = 0.219 \text{ (3 dp)}$$

17.08 & 17.09 Applications of quadratic equations

Example

An architect is designing a gym, which will include a basketball court measuring 29 m by 15 m. The gym is to be rectangular, with walls that are the same distance, x metres, from each edge of the basketball court. In order to fit seating, etc., the area outside the court is to be three times the area of the court itself. Calculate the value of x. Show any equations that you form in order to solve this problem.



Answer

The area of the basketball court is $29 \times 15 = 435$ m^2.

The area surrounding the court is $3 \times 435 = 1305$ m^2.

Therefore, the area of the whole gym is $435 + 1305 = 1740$ m^2.

In terms of x, the area of the whole gym is $(2x + 29) \times (2x + 15)$.

Therefore, x will be the solution to $(2x + 29)(2x + 15) = 1740$.

This quadratic equation simplifies to $4x^2 + 88x - 1305 = 0$.

How do we solve this equation?

1 It doesn't factorise.

2 We could complete the square. Divide through by 4 first.

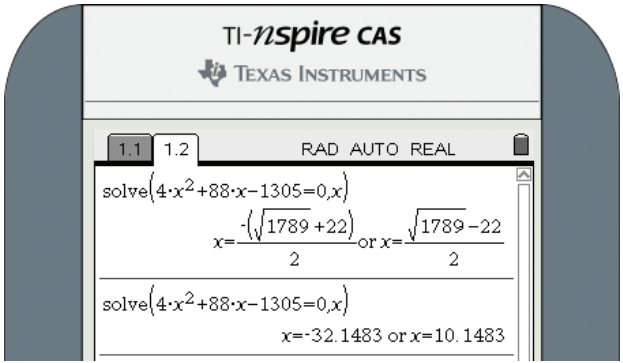
3 We could use the quadratic formula.

4 We could use an equation-solving package or CAS calculator.

The two solutions are $^{-}32.15$ and 10.15.

x is a length so discard the negative solution.

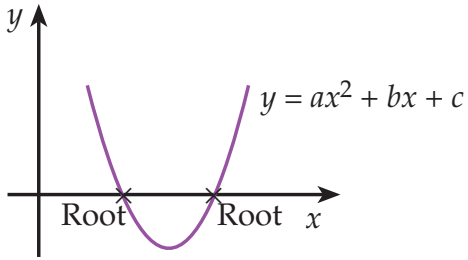
Each wall is 10.15 m from an edge of the court.



17.10 The discriminant of a quadratic equation

Nature of the roots of a quadratic equation

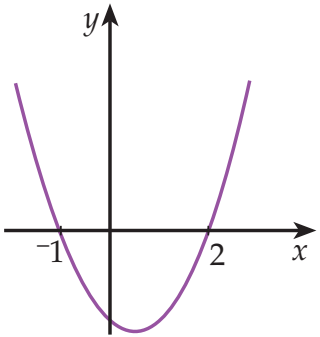
Quadratic expressions have **parabolas** for their graphs. The roots (solutions) of a quadratic equation in the form $ax^2 + bx + c = 0$ are given by the x-values where the parabola crosses the x-axis. This is because when a graph crosses the x-axis, the value of y is 0.



Example

Draw the graph of $y = (x + 1)(x - 2)$ and hence give the roots of the equation $(x + 1)(x - 2) = 0$.

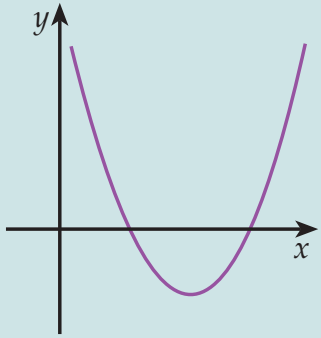
Answer



The x-intercepts are $^{-}1$ and 2. These are the roots of this quadratic equation.

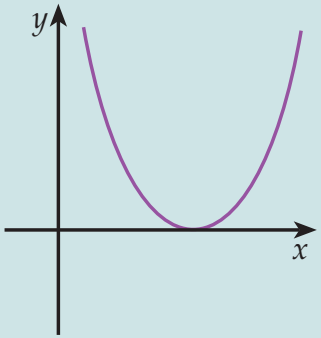
There are three possibilities when drawing a parabola for a quadratic expression.

1 The parabola may cross the x-axis at two points.



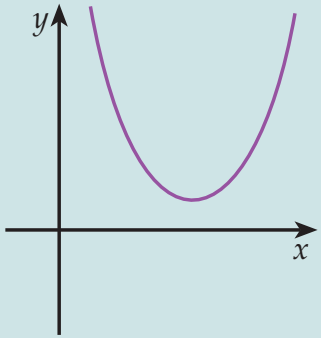
The associated quadratic equation is said to have **two real roots**.

2 The parabola may only touch the x-axis at a *single* point.



The associated quadratic equation is said to have **one repeated real root**.

3 The parabola may not cross the x-axis at all.



The associated quadratic equation is said to have **no real roots**.

The discriminant

The **discriminant** of a quadratic equation $ax^2 + bx + c = 0$ is the number: $b^2 - 4ac$. The discriminant is important because it is the number under the square root sign in the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Again, there are three cases, which tell us the number of real roots.

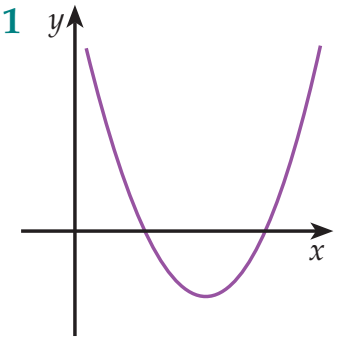
1 $b^2 - 4ac > 0$ The number under the square root sign is positive, so there are **two real roots**.

2 $b^2 - 4ac = 0$ The square root part of the formula is zero, so **one repeated root** for x is $\frac{-b}{2a}$.

3 $b^2 - 4ac < 0$ The number under the square root sign is negative. It is not possible to calculate the square root of a negative number within the real numbers, $\mathbb{R}$, so there are **no real roots**.

Combining these discriminant results with the parabola graphs, we have:

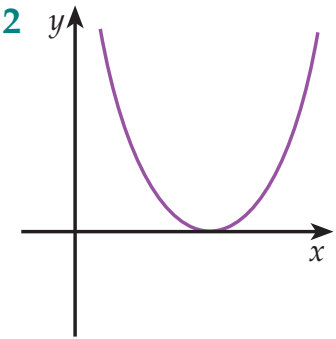
1



$b^2 - 4ac > 0$

Two real roots

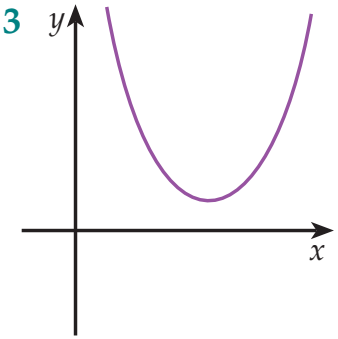
2



$b^2 - 4ac = 0$

One repeated root

3



$b^2 - 4ac < 0$

No real root

Example

Calculate the value of the discriminant of the quadratic equation $3x^2 - 4x + 5 = 0$, and hence state the number of roots for this quadratic equation.

Answer

For the equation $3x^2 - 4x + 5 = 0$,

$a = 3$, $b = {}^{-}4$ and $c = 5$.

$$b^2 - 4ac = ({}^{-}4)^2 - 4 \times 3 \times 5$$
$$= 16 - 60 = {}^{-}44$$

Here the discriminant is negative (< 0) and so this quadratic equation has *no real roots*.

17.11 Properties of the discriminant

Example

The equation $3x^2 - 4x + q = 0$ has two real roots. Find the range of possible values for q.

Answer

To have two real roots, we require $b^2 - 4ac > 0$.

For this equation:

$$(^-4)^2 - 4 \times 3 \times q > 0$$

$$16 - 12q > 0$$

$$^-12q > ^-16$$

$$q < \frac{^-16}{^-12}$$

$$q < \frac{4}{3}$$

(Note: the inequality reverses on division by a negative number)

18.01 Using the power button on your calculator

We represent repeated multiplication by using **powers**. The number being multiplied by itself repeatedly is called the **base**, and the number of bases in the product is called the **index**.
We write base$^{\text{index}}$. For example, $25 \times 25 \times 25 \times 25 = 25^4$.
Powers can be evaluated on a calculator: $25^4 = 390\,625$.
Note: the plural of 'index' is 'indices'.

Scientific calculators have a button that calculates powers.
On most models, this will be the x^y key.

18.02 Negative powers

What does a negative index mean? How do we interpret expressions such as x^{-3} or 2^{-4}?

Consider the division of $\frac{x^4}{x^6}$. If written out in full, this represents:

$$\frac{x \times x \times x \times x}{x \times x \times x \times x \times x \times x} = \frac{1}{x^2}.$$

But, if we follow the division rule for powers and subtract the indices, we get x^{-2}.

A power with a negative index is the same as the reciprocal of the same power with a positive index:

$$x^{-n} = \frac{1}{x^n}$$

Example

Write the following with a positive index.

a x^{-3} **b** $2x^{-1}$

Answer

a $x^{-3} = \frac{1}{x^3}$ **b** $2x^{-1} = 2 \times \frac{1}{x} = \frac{2}{x}$

Example

Write $\left(\frac{2}{3}\right)^{-2}$ as a fraction.

Answer

Note: the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

$\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^{2} = \frac{9}{4}$

Example

Evaluate the following.

a 2^{-4} **b** $(0.5)^{-3}$

Answer

a $2^{-4} = \frac{1}{2^4} = \frac{1}{16}$

b $(0.5)^{-3} = \frac{1}{(0.5)^3} = \frac{1}{0.125} = 8$

Alternatively:

$(0.5)^{-3} = \left(\frac{1}{2}\right)^{-3} = 2^3 = 8$

18.03 The rules of indices

Expressions with negative indices follow the same rules as expressions with positive indices.

- **Multiplication**: When *multiplying* numbers with the same base in power form, *add* the indices.
- **Division**: When *dividing* numbers with the same base in power form, *subtract* the indices.
- **Powers of powers**: When a base to a given index is raised to another index, simplify by *multiplying* the indices.

Example

Simplify the following.

a $3x^{-2} \times 4x^{-3}$

b $(xy^2)^{-4}$

Write both answers with positive indices only.

Answer

a $3x^{-2} \times 4x^{-3} = 12x^{-5} = \frac{12}{x^5}$

b $(xy^2)^{-4} = x^{-4}y^{-8} = \frac{1}{x^4y^8}$

18.04 Fractional indices

What does a fractional index mean? How do we interpret expressions such as $x^{\frac{1}{3}}$? What is the value of $64^{\frac{1}{2}}$? What meaning do we give to $8^{\frac{3}{2}}$?

Fractional indices are used to express *roots* of numbers. Recall that the square root of a number has the property that, when multiplied by itself, it gives the number concerned. For example:

$\sqrt{y} \times \sqrt{y} = y \qquad \sqrt{5} \times \sqrt{5} = 5$ etc.

Now consider equivalent expressions with indices. For example:

$y^{\frac{1}{2}} \times y^{\frac{1}{2}} = y^1 = y$ (adding indices)

$5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^1 = 5$ (adding indices)

Clearly, the power of $\frac{1}{2}$ is the same as 'square root'. In the same way, a power of $\frac{1}{3}$ gives 'cube roots', and so on. Collectively, square roots, cube roots, fourth roots, etc. are called **surds**.

Note: there is a special key for roots on a calculator. Depending on the make and model of the calculator, it will either be $\sqrt[x]{y}$ or $x^{1/y}$. Using this key often needs the 'inverse' or 'F' key pressed first.

Example

Write $x^{\frac{1}{5}}$ in surd form.

Answer

$x^{\frac{1}{5}} = \sqrt[5]{x}$

Example

Evaluate $64^{\frac{1}{3}}$.

Answer

$64^{\frac{1}{3}} = \sqrt[3]{64} = 4$

Example

Simplify $(64x^6)^{\frac{1}{2}}$.

Answer

$(64x^6)^{\frac{1}{2}} = 8x^3$

Example

Write $\sqrt[4]{x}$ in index form.

Answer

$\sqrt[4]{x} = x^{\frac{1}{4}}$

Powers and roots can be combined in the same index.

For example, $x^{\frac{2}{3}}$ means 'the cube root of x squared'.

In general: $x^{\frac{p}{q}} = \sqrt[q]{x^p}$

Example

Evaluate $16^{\frac{3}{2}}$.

Answer

$16^{\frac{3}{2}} = \left(16^{\frac{1}{2}}\right)^3 = 4^3 = 64$

Example

Write $\sqrt{x^5}$ in index form.

Answer

$\sqrt{x^5} = (x^5)^{\frac{1}{2}} = x^{\frac{5}{2}}$

(on multiplying the indices)

18.05 Fractional and negative indices

It is possible to use both negative indices and fractional indices at the same time. An approach that works well is to first deal with the negative index by taking the reciprocal of the base and changing the sign of the index at the same time.

Example

Write $x^{\frac{-2}{3}}$ in surd form.

Answer

$$x^{\frac{-2}{3}} = \frac{1}{x^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{x^2}}$$

Example

Evaluate the following.

a $625^{\frac{-1}{4}}$

b $\left(\frac{4}{9}\right)^{\frac{-3}{2}}$

Answer

a $625^{\frac{-1}{4}} = \frac{1}{625^{\frac{1}{4}}} = \frac{1}{\sqrt[4]{625}} = \frac{1}{5}$

b $\left(\frac{4}{9}\right)^{\frac{-3}{2}} = \left(\frac{9}{4}\right)^{\frac{3}{2}} = \left(\sqrt{\frac{9}{4}}\right)^{3} = \left(\frac{3}{2}\right)^{3} = \frac{27}{8}$

18.06 Expressing numbers as powers of the same base

Many numbers can be written quite simply as a power of another number. For example, 9 can be written as a power of 3, i.e. 3^2. $\frac{1}{27}$ can also be written as a power of 3, i.e. 3^{-3}.

A useful approach when simplifying expressions where some bases are powers of another base is to express all terms as powers of the *smaller* number.

Example

Simplify $\frac{4^{3n}}{2^{5n}}$.

Answer

First note that $4 = 2^2$. Therefore, replace 4 by its equivalent 2^2.

$$\frac{4^{3n}}{2^{5n}} = \frac{(2^2)^{3n}}{2^{5n}} = \frac{2^{6n}}{2^{5n}} = 2^{6n-5n} = 2^n$$

19.01 Formal definition of logarithms

If $b^p = q$ then $\log_b(q) = p$

Index form Log form

b is called the **base**, p is called the **logarithm** and q is the **number**.

We say $\log_b(q)$ as 'log of q to base b'.

Example

Write a log statement equivalent to $3^4 = 81$.

Answer

$3^4 = 81$ is the same as $\log_3(81) = 4$.

Example

Write the statement $\log_2(128) = 7$ in index form.

Answer

$\log_2(128) = 7$ is the same as $2^7 = 128$.

In some cases, we can use our knowledge of powers of numbers to find logarithms of whole-number bases and to solve simple equations.

Example

Work out the following.

a $\log_5(125)$

b $\log_{10}(100)$

Answer

a 5 to what power gives 125?

$5^3 = 125$. Therefore, $\log_5(125) = 3$.

b What power of 10 gives 100?

$10^2 = 100$. Therefore, $\log_{10}(100) = 2$.

Example

Solve the equation $\log_3(x) = 2$.

Answer

This equation is equivalent to

$3^2 = x$.

i.e. $x = 3^2 = 9$

Example

Find the value of p when $\log_p(64) = 6$.

Answer

$\log_p(64) = 6$ is a solution of the equation

$p^6 = 64$.

To solve this equation, take the 'sixth root' of both sides.

Note: $(^-2)^6 = 64$, but p must be positive (bases of logarithms are positive numbers) so we take

$p = 2$, not $^-2$.

19.02 Properties of logarithms – 1

KEY POINTS

Logarithms obey three rules.

1 $\log(ab) = \log(a) + \log(b)$

When multiplying numbers, *add* their logarithms.

2 $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$

When dividing numbers, *subtract* their logarithms.

3 $\log(a^n) = n\log(a)$

When raising a number to a power, *multiply* the logarithm by that power.

Example

Simplify $\log(3) + \log(5)$.

Answer

From rule **1** (on page 286),

$\log(3) + \log(5) = \log(3 \times 5) = \log(15)$.

Example

Simplify $\log(12) - \log(2)$.

Answer

From rule **2** (on page 286),

$\log(12) - \log(2) = \log\left(\frac{12}{2}\right) = \log(6)$.

Example

Write $3\log(2)$ as the log of a single number.

Answer

$3\log(2) = \log(2^3) = \log(8)$

19.02 Properties of logarithms – 2

Harder examples involve using more than one property at a time.

Example

Simplify $4\log(2) + \log(3) - \log(48)$.

Answer

$$\begin{aligned} 4\log(2) + \log(3) - \log(48) &= \log(2^4) + \log(3) - \log(48) \\ &= \log(16) + \log(3) - \log(48) \\ &= \log(48) - \log(48) \\ &= 0 \end{aligned}$$

Example

Simplify $\dfrac{\log(64)}{\log(4)}$.

Answer

$$\frac{\log(64)}{\log(4)} = \frac{\log(4^3)}{\log(4)}$$

$$= \frac{3\log(4)}{\log(4)}$$

$$= 3$$

19.03 Index equations

Logarithms can be used to solve 'index' equations; these equations have x in the exponent or index. The approach is to take logarithms of both sides.

Example

Solve the equation $2^x = 17$.

Answer

$$2^x = 17$$
$$\log(2^x) = \log(17)$$
$$x\log(2) = \log(17)$$
$$x = \frac{\log(17)}{\log(2)} = \frac{1.2304}{0.3010} = 4.087$$

The 4.087 here was calculated using the full calculator values for log(17) and log(2). But if 1.2304 is divided by 0.3010, then the result is 4.088.

This shows that it is best not to round an answer until the *end* of a problem.

Example

Evaluate $\log_4(32)$.

Answer

Let $x = \log_4(32)$.

This is equivalent to solving the equation $4^x = 32$.

$$
\begin{aligned}
4^x &= 32 \\
\log(4^x) &= \log(32) \quad \text{(taking logs of both sides)} \\
x\log(4) &= \log(32) \\
x &= \frac{\log(32)}{\log(4)} = \frac{\log(2^5)}{\log(2^2)} = \frac{5\log(2)}{2\log(2)} = \frac{5}{2}
\end{aligned}
$$

Alternatively, $\log(32) \div \log(4) = 2.5$ on a calculator.

i.e. $\log_4(32) = \frac{5}{2}$

19.04 Applications of index equations

Example

Moore's Law (see Starter above) can also be expressed as an index equation:

number in the future = number now $\times$ 1.6^t, where t is the number of years from now.

Solve the equation $1.6^t = 10$. Explain what the solution represents in the context of Moore's Law.

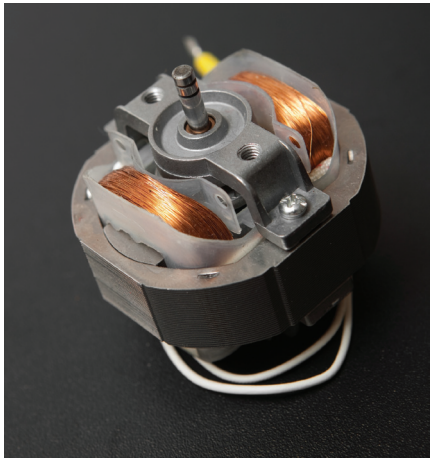
Answer

$$\begin{aligned} (1.6)^t &= 10 \\ \log(1.6^t) &= \log(10) \\ t\log(1.6) &= \log(10) \\ t &= \frac{\log(10)}{\log(1.6)} = 4.899 \text{ (4 sf)} \end{aligned}$$

It will take 4.9 years (i.e. slightly less than 5 years) for the number of transistors to be 10 times the number now.

Example

An electric motor loses efficiency at a steady rate of 4% per year as its components deteriorate. How many years does it take for the motor to decline from a rating of 2500 watts to 2160 watts?



Answer

Use the rule (new amount) = (old amount) × (multiplier)^n where n is the number of time periods.

The multiplier that gives a 4% decrease is 0.96.

Therefore:

$$2160 = 2500 \times (0.96)^n$$

$$(0.96)^n = \frac{2160}{2500} = 0.864$$

Take logs of both sides:

$$\log(0.96)^n = \log(0.864)$$

$$n \log(0.96) = \log(0.864)$$

$$n = \frac{\log(0.864)}{\log(0.96)}$$

$$= \frac{^-0.063\,486}{^-0.017\,729}$$

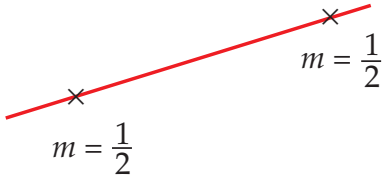
$$= 3.58 \text{ (2 dp)}$$

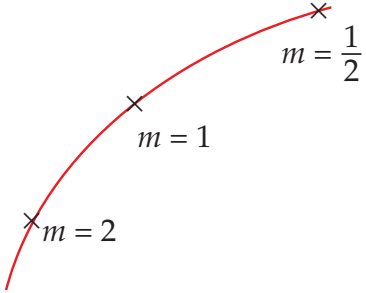
An appropriate answer would be about $3\frac{1}{2}$ years, or somewhere in the fourth year.

20.01 Gradients of curves

The gradient, or slope, of a **line** is a number that describes how steep the line is. On a line, the gradient is the same regardless of where you measure it.

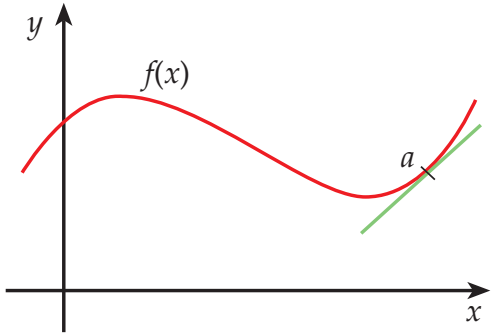
However, on a **curve**, *the steepness is changing all the time.*





We define the gradient of a curve *at a point* to be the gradient of the tangent drawn to the curve at that point.

The gradient of the curve below at a is the same as the gradient of the **tangent** at a.



In general, the rule for the equation of the curve is some function, $f(x)$.

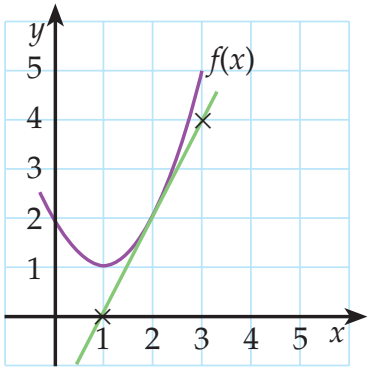
The gradient of this curve would be different if we chose another point.

Although the gradient of the curve is changing all the time, we will discover soon that a special formula, called the **derived function** or **derivative** of $f(x)$, gives the gradient at *any* point. The symbol for the derived function is $f'(x)$.

Example

The graph of a function $f(x)$ is shown here.

A tangent is drawn at $x = 2$. Use the graph to evaluate $f'(2)$, the derivative of $f(x)$ at $x = 2$.



Answer

The tangent passes through the points
(1, 0) and (3, 4).

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{4 - 0}{3 - 1}$$
$$= \frac{4}{2} = 2$$

20.02 & 20.03 Derivatives of polynomials

There is a rule that gives derived functions for powers of x. This means we don't have to calculate gradients all the time!

The rule for differentiation (finding derived functions) of powers of x is:

$$f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1}$$

The rule says, 'Take the power of x out to the front, and subtract 1 from the power.' In other words, multiply by the index, and drop the index by 1.

Example

Differentiate the following.

a x^4 **b** x **c** 6

Answer

a $4x^3$

b Note: $x = x^1$.
This differentiates to $1x^{1-1} = 1x^0 = 1$

c $6 = 6x^0$.
x^0 differentiates to $0x^{-1} = 0$.

Example **c** shows that constant terms have a derived function of 0. This is what we would expect: if the graph of $f(x) = 6$ were drawn (i.e. $y = 6$), it would be a horizontal line with a gradient of 0.

Multiples of powers of x

Example

Differentiate $y = 5x^3 - 2x$.

Answer

$$\frac{dy}{dx} = 5 \times 3x^2 - 2$$
$$= 15x^2 - 2$$

When there is a number in front of (multiplying) the power, that number stays there when differentiating:

$$f(x) = ax^n \quad \Rightarrow \quad f'(x) = anx^{n-1}$$

KEY POINTS

1. The above example shows that when a function includes several terms added or subtracted together, we can differentiate term by term.
2. We call the process in the example above **differentiating by sight** – that is, we use the rule for differentiating powers of x without going back to investigating gradients.

20.04 Differentiating simple products and quotients

The rules for differentiating only work on simple polynomials, with terms given one by one. When differentiating a product, or a quotient, expand or simplify first if possible to express as a simple polynomial before differentiating.

Example

Differentiate $y = x^2(2x^3 - 5)$.

Answer

Expand first:

$y = x^2(2x^3 - 5)$

$y = 2x^5 - 5x^2$

Then differentiate:

$$\frac{dy}{dx} = 2 \times 5x^4 - 5 \times 2x$$

$$= 10x^4 - 10x$$

Example

Differentiate $f(x) = \dfrac{2x^3 + 5x^2}{x}$.

Answer

Simplify by dividing first:

$$f(x) = \frac{2x^3 + 5x^2}{x}$$
$$= \frac{2x^3}{x} + \frac{5x^2}{x}$$
$$= 2x^2 + 5x$$

Then differentiate:

$$f'(x) = 2 \times 2x + 5$$
$$= 4x + 5$$

20.05 Substitution into derived functions

In the previous exercises, we covered the process of differentiation for various functions. The function we derive this way gives a formula for finding the gradient. To actually evaluate the gradient, we need to substitute a value for x.

Example

$f(x) = x^3 - 4x^2 + 5x - 6$. Calculate $f'(2)$.

Answer

$$\begin{aligned} f'(x) &= 3x^2 - 8x + 5 \\ f'(2) &= 3 \times 2^2 - 8 \times 2 + 5 \\ &= 12 - 16 + 5 \\ &= 1 \end{aligned}$$

Example

$f(x) = x^2 - 4x + 7$.

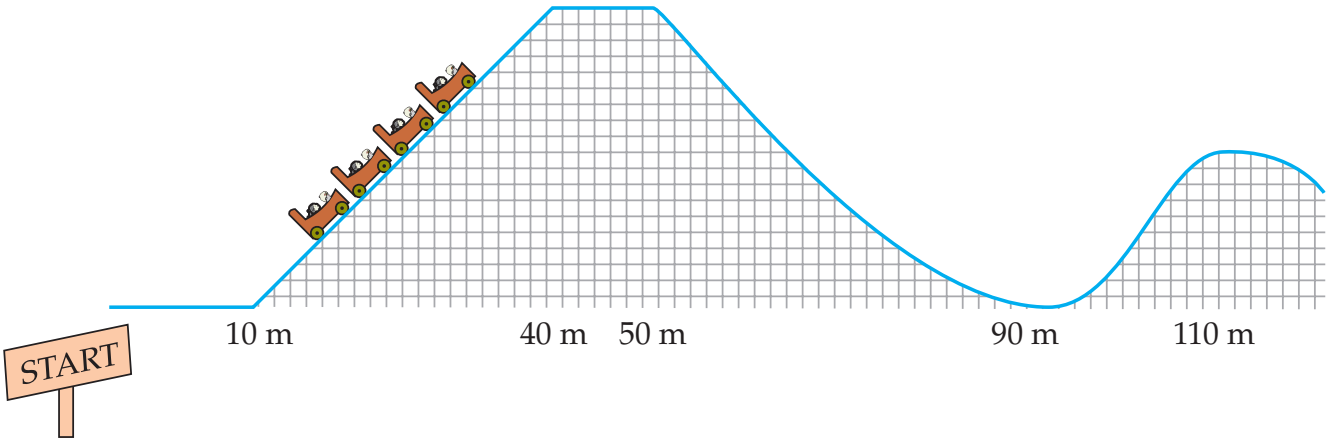
Solve the equation $f'(x) = 6$.

Answer

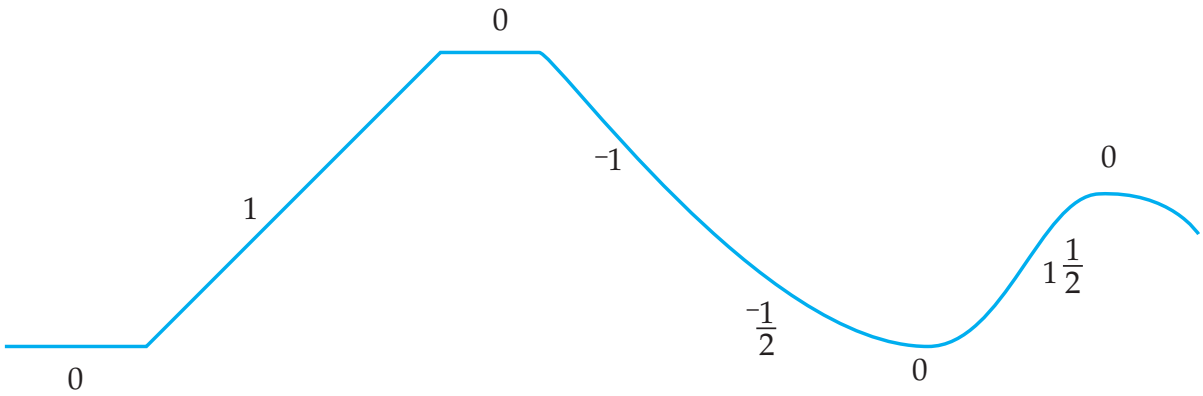
$$\begin{aligned} f'(x) &= 2x - 4 \\ 2x - 4 &= 6 \\ 2x &= 10 \\ x &= 5 \end{aligned}$$

20.06 Sketching derived functions

The diagram shows the cross-section of a roller coaster over the first 110 m from its start.



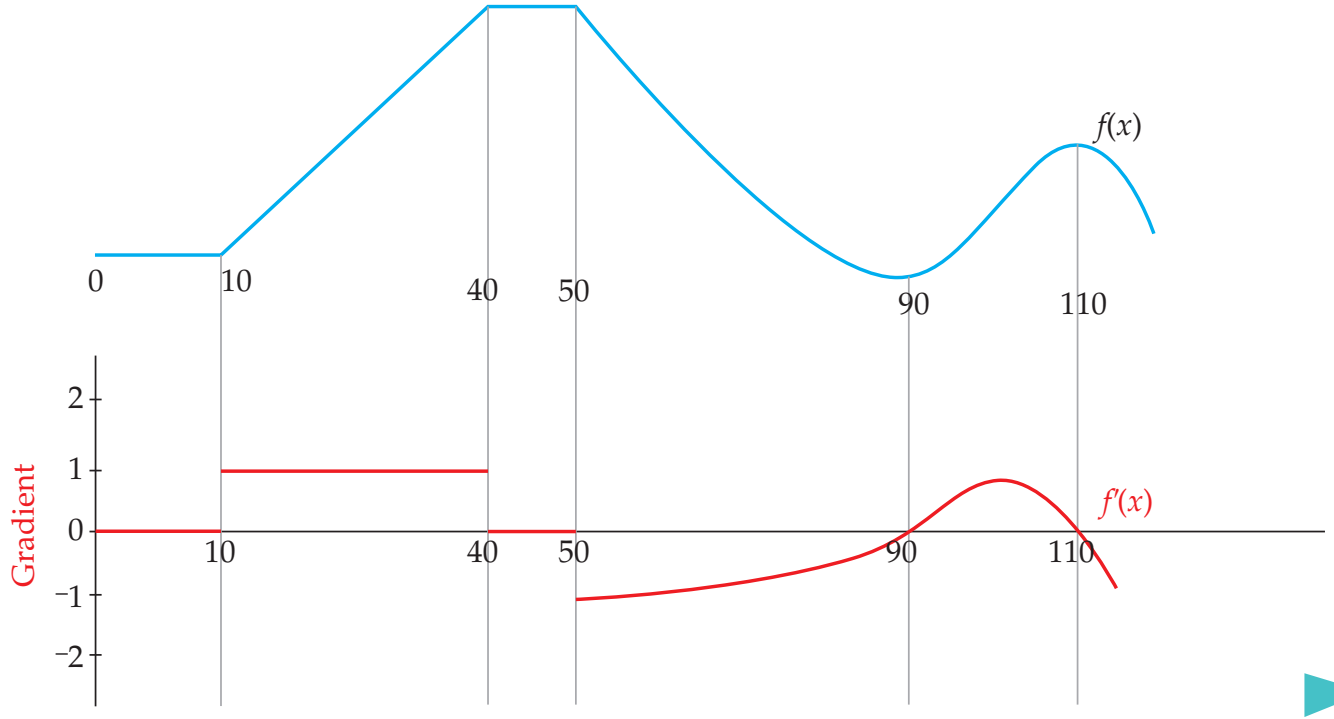
Concentrate on the steepness that the riders are experiencing as they move along the track.



- When the track is level, the gradient is 0.
- Going upwards, at a 45° angle, the gradient is 1.
- The track is then level again, with a gradient of 0.
- Going down, the gradient is steep ($^{-}1$), then less steep $\left(\frac{^{-}1}{2}\right)$.
- The gradient is briefly 0 at the bottom of the curve.
- Going up again, the gradient increases to about $1\frac{1}{2}$.

- Then the steepness goes down from about $1\frac{1}{2}$ until the track is level (gradient of 0).

This graph shows the **gradient** (drawn in red) at various distances from the start.



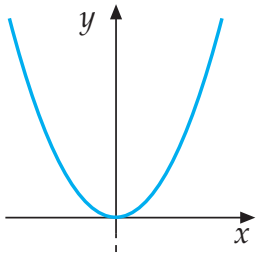
Notice that:

- $f'(x)$ is 0 between 0 and 10, and also between 40 and 50 – this is where the track is level
- when the roller coaster is going up (height **increasing**), the gradient function $f'(x)$ is **positive**
- when the roller coaster is going down (height **decreasing**), the gradient function $f'(x)$ is **negative**.

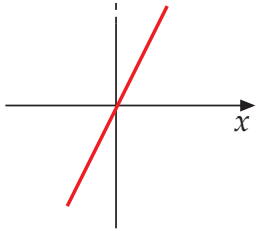
The best place to draw the derived function graph, or gradient graph, is immediately below the function graph.

Example

$f(x) = x^2$

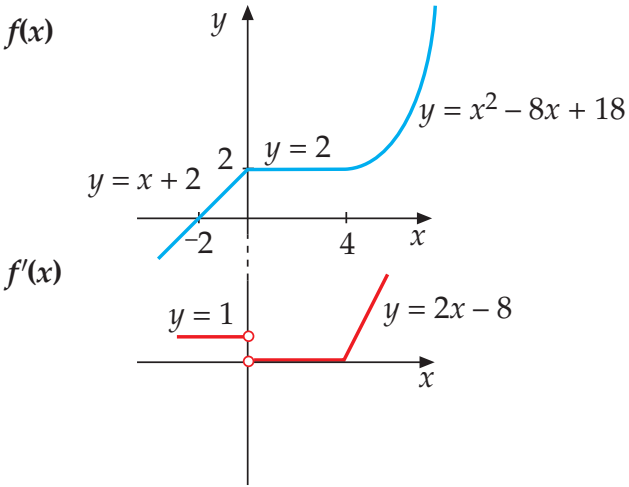


$f'(x) = 2x$



The equation of the function need not be known to sketch the graph of the derived function. Just examine the graph of the original $f(x)$ function, and read off or estimate the gradient at various x-values. Then transfer the value of the gradient to the corresponding x-value below.

Example



Proceeding from left to right:

- the first part of $f(x)$ has a gradient of 1 so, below this part of the function, we draw the graph of $y = 1$
- then, the horizontal portion of the graph of $f(x)$ clearly has gradient 0, so we draw the line $y = 0$ below this part
- finally, the parabolic portion on the right will have gradient $2x - 8$, so we draw a line with gradient 2

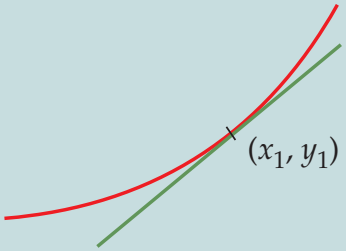
Note that the gradient is undefined when $x = 0$, so the derived function graph has no value at $x = 0$. The empty (or open) circles at $x = 0$ show that 0 is excluded from the domain of $f(x)$.

Take a look at some applets that draw the graph of the derived function for various functions. Each applet shows how the gradient changes as you move along a graph. There are links to these applets from the *Theta Mathematics* links page at www.mathematics.co.nz.

21.01 Tangents to curves

As we saw earlier, the process of differentiation results in a 'derived function' that gives a formula for the gradient at any point on a curve. Given a function $f(x)$, we substitute the x-co-ordinate into $f'(x)$ to find the gradient.

The gradient at a point on a curve is the same as the gradient of the tangent at that point.



Example

Write the gradient of the curve $y = x^2$ at the point $(3, 9)$.

Answer

$f'(x) = 2x$

$f'(3) = 6$

That is, the gradient is 6.

To determine the gradient at a point, we only need to substitute the *x*-value into the derived function. The *y*-co-ordinate is not used.

21.02 Finding the point where the gradient has a given value

Given the value of the gradient, it is possible to form an equation using the *derived* function. This equation is solved for x (giving the x-co-ordinate, of course), and then the corresponding y-value can be calculated by substituting the x-value just found into the *original* function.

Example

What are the co-ordinates of the point on the curve given by $y = x^2 - 2x - 5$ where the gradient is 6?

Answer

At the required point, the gradient is 6. That is:

$$\frac{dy}{dx} = 6$$
$$2x - 2 = 6$$
$$2x = 8$$
$$x = 4$$

The corresponding y-value is found by substituting into the original function, that is:

$$y = x^2 - 2x - 5$$
$$y = 4^2 - 2 \times 4 - 5$$
$$= 16 - 8 - 5$$
$$= 3$$

The co-ordinates are (4, 3).

21.03 The equation of a tangent to a curve

To obtain the equation of the tangent to a curve, given the co-ordinates (x_1, y_1) of a point on the curve:

1. differentiate to obtain the derived function
2. substitute x_1 into the derived function to find the tangent *gradient* (m)
3. then use the **point–gradient equation** $y - y_1 = m(x - x_1)$ for the tangent *equation*.

Example

Write the equation of the tangent to $y = x^2 - 6x - 2$ at the point $(7, 5)$.

Answer

$f'(x) = 2x - 6$

$f'(7) = 2 \times 7 - 6 = 8$

That is, the gradient is 8.

Now use the point–gradient equation $y - y_1 = m(x - x_1)$.

Note: $m = 8$, $x_1 = 7$ and $y_1 = 5$.

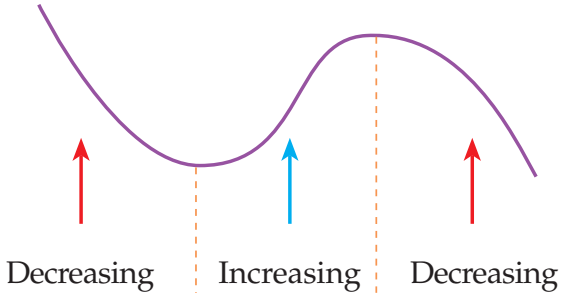
Equation of the tangent is:

$$y - 5 = 8(x - 7)$$
$$y - 5 = 8x - 56$$
$$y = 8x - 51$$

or, writing in the form $ax + by + c = 0$: $8x - y - 51 = 0$.

21.04 Increasing and decreasing functions

- A function is said to be **increasing** when it has a *positive* gradient.
- A function is said to be **decreasing** when it has a *negative* gradient.



We can use calculus to work out the x-values for which a function is increasing or decreasing:

- $f(x)$ is increasing when $f'(x) > 0$
- $f(x)$ is decreasing when $f'(x) < 0$.

Example

For what values of x is the graph of $y = x^2 - 4x + 5$ decreasing?

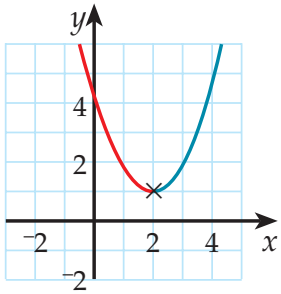
Answer

$$\frac{dy}{dx} = 2x - 4 < 0 \text{ for a decreasing graph.}$$

$$2x - 4 < 0$$

$$2x < 4$$

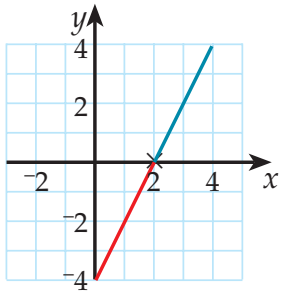
$$x < 2$$



Original function

- Increasing when $x > 2$
- Decreasing when $x < 2$
- Turning point when $x = 2$

The relationship between the original function and the derived function can be shown by comparing their graphs.



Derived function
- Positive when $x > 2$
- Negative when $x < 2$
- Zero when $x = 2$

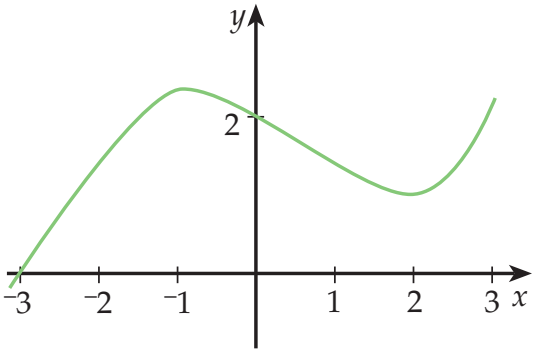
- For x-values where the original function is increasing, the derived function is positive (has positive y-values so is above the x-axis).
- For x-values where the original function is decreasing, the derived function is negative (has negative y-values so is below the x-axis).

Example

Find the set of x-values for which the graph drawn here is increasing.

Answer

$x < -1$ or $x > 2$.

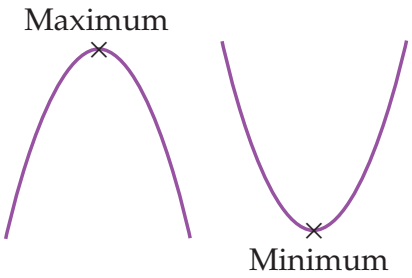


Turning points

Suppose a function has a smooth graph (without sharp corners).

A **turning point** on the graph of a function is the point where the function changes from being increasing to being decreasing, or from decreasing to increasing.

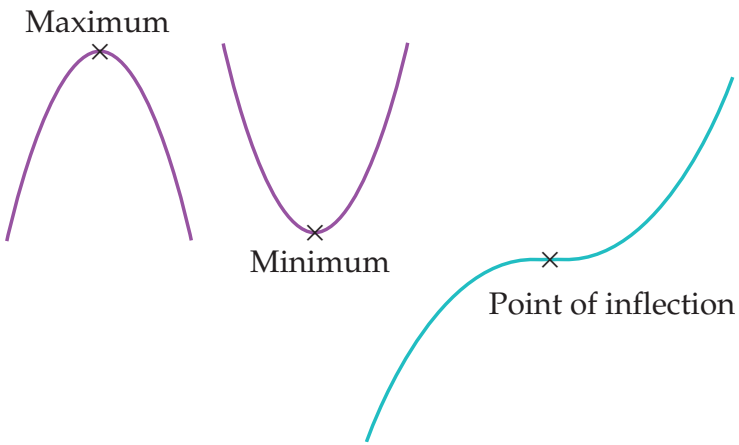
This means that, at a turning point, the gradient is zero, i.e. $f'(x) = 0$.



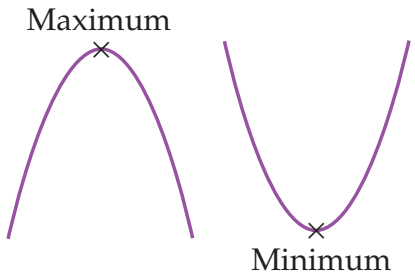
There are two kinds of turning point: a **maximum point** (plural maxima) and a **minimum point** (plural minima). At both points, $f'(x) = 0$.

Stationary points

The definition of a **stationary point** is a point where $f'(x) = 0$. Stationary points include both maxima and minima, but also another kind of point – a **point of inflection**.



21.05 Maximum and minimum points



There are two types of turning point on a graph – maximum and minimum. At each type of point, the gradient is 0. To find the co-ordinates of a turning point:

- solve the equation $f'(x) = 0$; the solution to this equation gives the x-co-ordinate
- substitute the x-value into the original function $f(x)$ to obtain the y-co-ordinate.

You are often asked to say whether a minimum or maximum point has been obtained. Methods for deciding the nature of the point you have found include the following two.

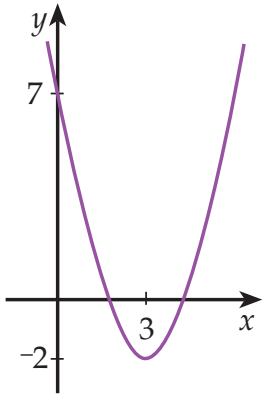
1 Draw the graph – the type of point should be obvious.

2 Evaluate the function at points just above and just below x, the turning point value.

- If these answers are greater than $f(x)$, then a minimum has been found.
- If these answers are less than $f(x)$, then a maximum has been found.

Example

Find the co-ordinates of the turning point on the graph of $y = x^2 - 6x + 7$. Determine the nature of the turning point.



Answer

$\frac{dy}{dx} = 2x - 6 = 0$ for a turning point:

$$2x - 6 = 0$$
$$2x = 6$$
$$x = 3$$

To find the y-co-ordinate, substitute $x = 3$ into the *original* equation:

$y = 3^2 - 6 \times 3 + 7 = 9 - 18 + 7 = {}^{-}2$

The turning point is $(3, {}^{-}2)$.

As the graph of the function is a parabola that has branches up, the turning point is obviously a *minimum* point.

Example

Determine the co-ordinates of the turning points of the function given by $f(x) = x^3 - 3x + 2$.
State which point is a maximum and which is a minimum.

Answer

$f'(x) = 3x^2 - 3 = 0$ for turning points:

$$3x^2 = 3$$
$$x^2 = 1$$
$$x = \pm 1$$

The x-values that give turning points are 1 and $^-1$. To calculate the y-value at the turning points, substitute $x = 1$ and $x = {}^-1$ in turn into the original function.

$f(1) = 1^3 - 3 \times 1 + 2 = 1 - 3 + 2 = 0$

$f(^-1) = (^-1)^3 - 3 \times {}^-1 + 2 = {}^-1 + 3 + 2 = 4$

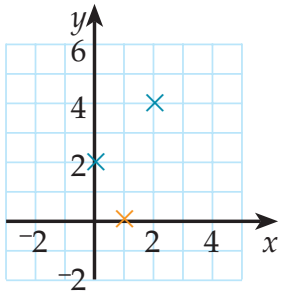
The two turning points are $(1, 0)$ and $(^-1, 4)$.

To determine the nature of the turning point (1, 0), evaluate $f(x)$ for values of x either side of 1:

$f(2) = 2^3 - 3 \times 2 + 2 = 8 - 6 + 2 = 4$

$f(0) = 0^3 - 3 \times 0 + 2 = 0 - 0 + 2 = 2$

The turning point is marked orange. Both of the nearby points are *higher* than (1, 0), which means there is a *minimum* point when $x = 1$, i.e. (1, 0) is a minimum.

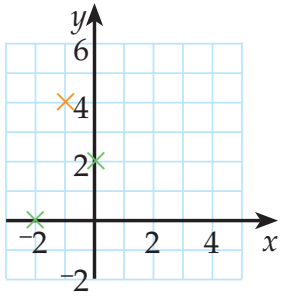


To determine the nature of the turning point $(^-1, 4)$, evaluate $f(x)$ for values of x either side of $^-1$:

$f(^-2) = (^-2)^3 - 3 \times {}^-2 + 2 = {}^-8 + 6 + 2 = 0$

$f(0) = 0^3 - 3 \times 0 + 2 = 0 - 0 + 2 = 2$

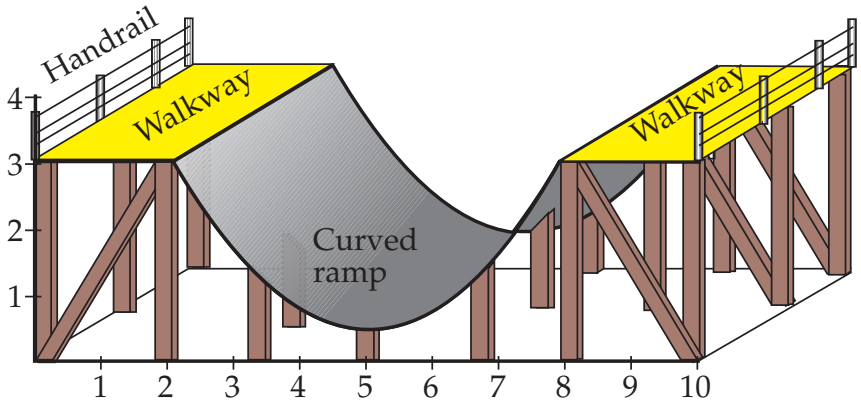
The turning point is marked orange. Both of the nearby points are *lower* than $(^-1, 4)$, which means there is a *maximum* point when $x = {}^-1$, i.e. $(^-1, 4)$ is a maximum.



21.06 Applications of calculus

The calculus we have used so far is helpful for working out optimum values in various simple models.

Example



The diagram shows the cross-section of a skateboard ramp. The ramp is supported by a number of wooden struts. There is a 2 m wide walkway at the top of the ramp on each side. The height, h, in metres above the ground of the curved part of the ramp can be represented by the equation $h = 0.3x^2 - 3x + 8$, for values of x between 2 and 8, where x is the horizontal distance in metres along the ground from one corner of the ramp.

a Calculate the height above the ground of the walkways.

b What is the height of the shortest strut shown in the diagram?

c Comment on whether there are some limitations to the answer in part **b**.

Answer

a The height of the walkways is the same as the height of the curved part of the ramp at its highest point. Substitute either $x = 2$ or $x = 8$ into the equation $h = 0.3x^2 - 3x + 8$:

$$h = 0.3 \times 2^2 - 3 \times 2 + 8$$
$$= 1.2 - 6 + 8$$
$$= 3.2 \text{ m}$$

b The information in the diagram implies we need to determine the *minimum* value of the function given by $h = 0.3x^2 - 3x + 8$.

Differentiate to find $h'(x)$:

$h'(x) = 0.6x - 3 = 0$ for turning points:

$$0.6x = 3$$

$$x = \frac{3}{0.6} = 5$$

In this case, it is obvious $x = 5$ gives a *minimum* point because the shape of the curve is given in the diagram.

The corresponding y-value is obtained by substituting $x = 5$ into the original function

$h = 0.3x^2 - 3x + 8$.

$$h = 0.3 \times 5^2 - 3 \times 5 + 8$$

$$h = 7.5 - 15 + 8$$

$$= 0.5 \text{ m}$$

c There are several limitations.

(1) The answer in part **b** ignores the thickness of the ramp surface.

(2) The strut may not be exactly underneath the centre of the curved ramp.

(3) The strut may not be vertical.

(4) The ground may not be level.

(5) The model itself is probably unrealistic. Most skateboard ramps have a cross-section that is flatter in the centre than a typical parabola.

22.01 The rule for anti-differentiation of polynomials

The opposite process to differentiation is called **anti-differentiation**. Anti-differentiation is also called **integration**.

- The symbol used to indicate integration is $\int$. This is called an integral symbol.
- We write dx after the expression to be integrated to show that x is the variable, and say 'integration with respect to x'.

We know that the derivative of, say, $f(x) = x^2$ is $f'(x) = 2x$. Reversing this, we say that the anti-derivative of $2x$ is the function x^2.

To find the anti-derivative, we are finding the function $f(x)$ which has a derivative $f'(x)$. That is, we are working backwards from the given derived function $f'(x)$ to find the original function $f(x)$.

In general, anti-derivatives are not unique. There are many functions, for example, that differentiate to $2x$:

x^2, $x^2 + 1$, $x^2 - 5$, $x^2 + 17$, and so on.

We write $\int 2x \, dx = x^2 + c$

where c is called the **constant of integration**.

To integrate any power of x, add 1 to the power, and divide by the new power.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

This rule works for all powers of x, except for $^{-}1$ because then we would have to divide by 0, which is undefined.

Example

Find $\int x^3 \, dx$.

Answer

$$\int x^3 \, dx = \frac{x^4}{4} + c$$

Multiples of powers of x are anti-differentiated in the same way.

$$\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c$$

Example

Find:

a $\int 6x^2 \, dx$

b $\int \frac{-2}{5}x^3 \, dx.$

Answer

a $\int 6x^2 \, dx = 6 \times \frac{x^3}{3} + c$

$= 2x^3 + c$

b $\int \frac{-2}{5}x^3 \, dx = \frac{-2}{5} \times \frac{x^4}{4} + c$

$= \frac{-1}{10}x^4 + c$

The rules for anti-differentiation can be extended to polynomials that involve several powers of x. For polynomials, anti-differentiate term by term.

Example

Find $\int (3x^2 + 6x - 5)\,dx$.

Answer

$$\int (3x^2 + 6x - 5)\,dx = 3 \times \frac{x^3}{3} + 6 \times \frac{x^2}{2} - 5 \times \frac{x^1}{1} + c$$
$$= x^3 + 3x^2 - 5x + c$$

Note that 5, the constant term, integrated to $5x$. This is because 5, written as a power of x, is $5x^0$.

22.02 Anti-differentiation of simple products and quotients

Expressions that include products or quotients in fraction form should be expanded or simplified by dividing *before* carrying out the anti-differentiation.

Example

What is the anti-derivative of $x^2(2x - 5)$?

Answer

$$\int x^2(2x - 5)\,dx = \int (2x^3 - 5x^2)\,dx$$

$$= 2 \times \frac{x^4}{4} - 5 \times \frac{x^3}{3} + c$$

$$= \frac{x^4}{2} - \frac{5x^3}{3} + c$$

Example

What is the anti-derivative of $\frac{4x^3 + 5x^2}{2x}$?

Answer

$$\int \frac{4x^3 + 5x^2}{2x}\,dx = \int \left(\frac{4x^3}{2x} + \frac{5x^2}{2x}\right) dx$$

$$= \int \left(2x^2 + \frac{5x}{2}\right) dx$$

$$= 2 \times \frac{x^3}{3} + \frac{5}{2} \times \frac{x^2}{2} + c$$

$$= \frac{2x^3}{3} + \frac{5x^2}{4} + c$$

22.03 The graphs of $f(x)$ and $f'(x)$

The rules for differentiation and anti-differentiation give us the relationship between a polynomial function and its derived function. This relationship involves the degrees of the two functions.

The degree of a polynomial is the highest power of x. For example, the degree of $f(x) = 5x^3 + x^2 - 7x + 1$ is 3.

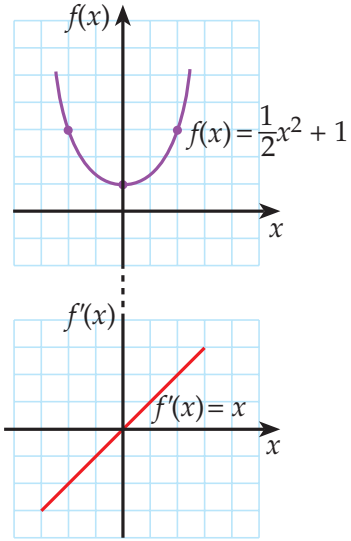
- The degree of a polynomial is one *more* than the degree of its derived function.
- The degree of the derived function is one *less* than the degree of the polynomial concerned.

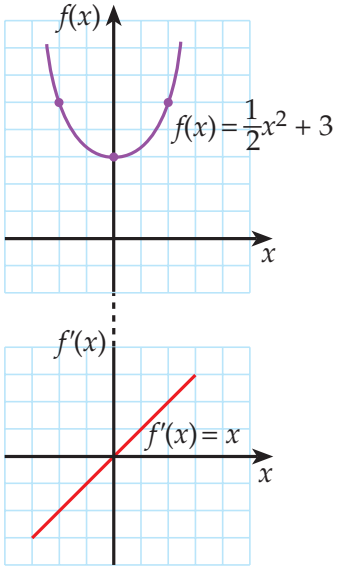
Type of polynomial		Graph		Example	
Function	Derived function	Function	Derived function	Function	Derived function
cubic	quadratic	cubic	parabola	$x^3 + 7x$	$3x^2 + 7$
quadratic	linear	parabola	line	$^-4x^2 + 5$	^-8x
linear	constant	line	horizontal line	$6x$	6

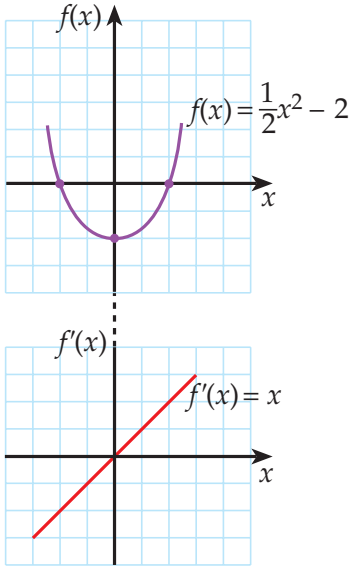
We already know how the gradient of the polynomial function gives the graph for the derived function. Working in the other direction (finding the graph of $f(x)$ given the graph of $f'(x)$) is not so straightforward. That is because there is no unique function that has a particular derived (gradient) function.

Example

All three of the functions below have the same derived (gradient) function.

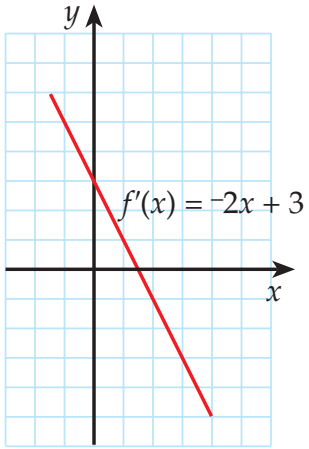




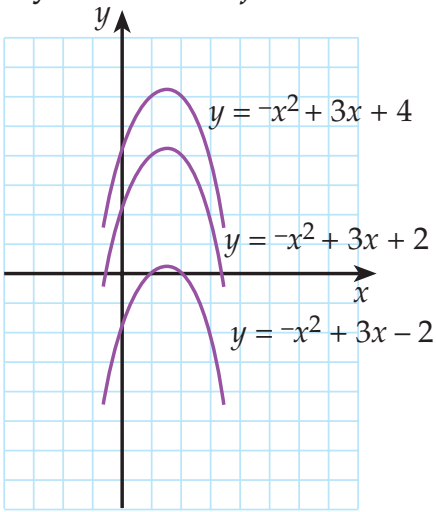


Given a particular graph for a derived function $f'(x)$, there is a **family of curves** that could be possible graphs for $f(x)$.

Gradient (derived) function



Family of curves for $f(x) = -x^2 + 3x + c$



The diagram above shows three possible positions for $f(x)$. There are many (an infinite number of) others.

22.04 Evaluating the constant of integration

So far, we have been integrating functions to obtain other functions as the answer, without explaining what the answer function represents.

One use of integration or anti-differentiation is to calculate the equation of a curve when we know the gradient function. If the co-ordinates of a point on the curve are known, we can use this information to calculate the value of c, the **constant of integration** we have been adding on to all anti-derivatives.

First, we will look at a very simple example of finding the constant of integration.

Example

$f'(x) = 2x - 9$ and $f(1) = 5$.
Find $f(x)$.

Answer

$$f(x) = \int (2x - 9)\, dx$$
$$= x^2 - 9x + c$$

Now, $f(1) = 5$ means that if 1 is substituted for x into $f(x) = x^2 - 9x + c$, we get a result of 5:

$$5 = 1^2 - 9 \times 1 + c$$
$$= 1 - 9 + c$$
$$c = 5 + 8 = 13$$
$$f(x) = x^2 - 9x + 13$$

Because we are working with equations of curves (in the form $y = \ldots$), it is often more convenient to express the gradient function in the form $\frac{dy}{dx} = \ldots$.

Example

A curve passes through the point $(^{-}2, 3)$ and has a gradient function given by: $\frac{dy}{dx} = 3x^2 - 8x + 1$.

Find the equation of the curve.

Answer

$$\frac{dy}{dx} = 3x^2 - 8x + 1$$

$$y = \int (3x^2 - 8x + 1)\, dx$$

$$= 3 \times \frac{x^3}{3} - 8 \times \frac{x^2}{2} + 1 \times x + c$$

$$= x^3 - 4x^2 + x + c$$

If $(^{-}2, 3)$ lies on the curve then $y = 3$ when $x = {}^{-}2$:

$$3 = (^{-}2)^3 - 4 \times (^{-}2)^2 + {}^{-}2 + c$$

$$= {}^{-}8 - 16 - 2 + c$$

$$= {}^{-}26 + c$$

$$c = 29$$

The equation of the curve is $y = x^3 - 4x^2 + x + 29$.

22.05 Definite integrals

So far, our anti-differentiation has had a *function* as the answer to the problem. In contrast, when we integrate between two values, called **limits of integration**, we are finding a *number* as the answer.

The **definite integral** is the number that results from substituting the two values (limits of integration) into the anti-derivative and then calculating the difference.

Example

Evaluate $\int_2^4 (2x - 3)\,dx$.

Answer

$$\begin{aligned}\int_2^4 (2x - 3)\,dx &= \left[x^2 - 3x\right]_2^4 \\ &= (4^2 - 3 \times 4) - (2^2 - 3 \times 2) \\ &= (16 - 12) - (4 - 6) \\ &= 4 - {}^{-}2 \\ &= 6\end{aligned}$$

Note that we substitute *both* of the limits of integration into the anti-derivative, and then subtract.

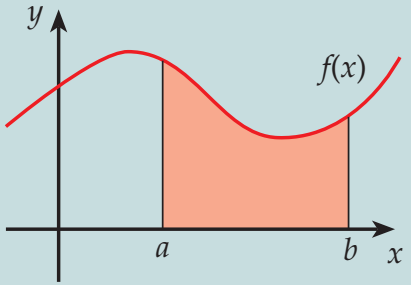
When doing this kind of *definite* integration, the constant of integration is eliminated when the two values are subtracted. The constant of integration is not usually written in because it has no effect on the final result.

22.07 Areas under curves

When we find a derived function, or derivative, we are finding the *gradient* of a curve. In a very similar way, the process of integration is equivalent to finding the *area* between a curve and the x-axis.

The usual convention when calculating the definite integral is to substitute the right-hand value, b, first, and then the left-hand value, a. This gives a positive answer if the area is above the x-axis. Areas below the x-axis turn out to have negative definite integrals when calculated.

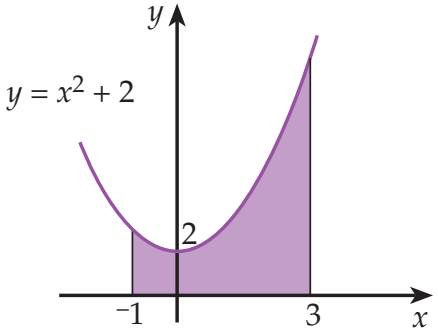
We define $\int_a^b f(x)\,\mathrm{d}x$ as the area between the curve of $f(x)$ and the x-axis, bounded by the vertical lines $x = b$ and $x = a$.



If the question specifically asks for *area*, the absolute value should be taken.

Example

Calculate the area between the curve $y = x^2 + 2$, the x-axis and the lines $x = {}^{-}1$ and $x = 3$.



Answer

$$\text{Area} = \int_{-1}^{3} (x^2 + 2)\, dx$$

$$= \left[\frac{x^3}{3} + 2x\right]_{-1}^{3}$$

$$= \left(\frac{3^3}{3} + 2 \times 3\right) - \left(\frac{(^{-}1)^3}{3} + 2 \times {}^{-}1\right)$$

$$= (9 + 6) - \left(\frac{^{-}1}{3} - 2\right)$$

$$= 15 - {}^{-}2\frac{1}{3}$$

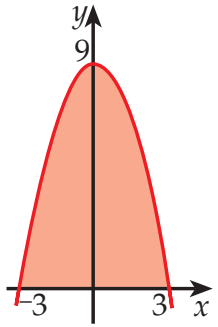
$$= 17\frac{1}{3} \text{ square units}$$

Example

Calculate the area between the curve $y = 9 - x^2$ and the x-axis.

Answer

First draw a graph to get a feeling for the required area.



The parabola intersects the x-axis when $y = 0$.

$$9 - x^2 = 0$$

$$(3 - x)(3 + x) = 0$$

$$x = 3 \text{ or } {}^{-}3$$

$$\text{Area} = \int_{-3}^{3} (9 - x^2)\, dx$$

$$= \left[9x - \frac{x^3}{3}\right]_{-3}^{3}$$

$$= (27 - 9) - ({}^{-}27 + 9)$$

$$= 18 - {}^{-}18$$

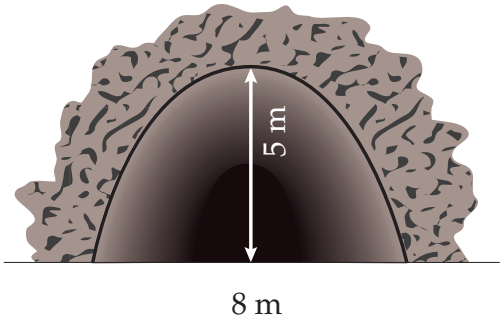
$$= 36 \text{ square units}$$

22.08 Applications of area

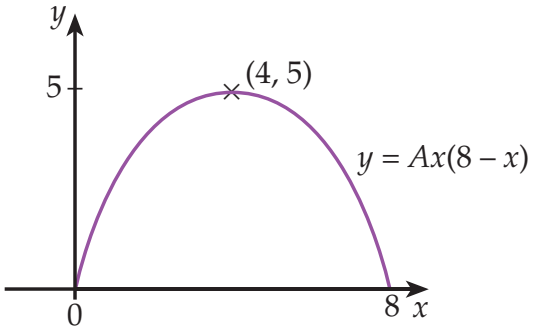
Example

A road tunnel that is 200 m long is constructed underneath a hill. The tunnel has a parabolic cross-section for its entire length. The tunnel is 8 m wide at road level and is 5 m high at the top. Estimate the volume of material (rock and soil) removed from the hill. Comment on the accuracy of your estimate.

Answer



The tunnel



The problem fitted to appropriate axes

First, determine the equation that gives the outline of the cross-section: it is an upside-down parabola with x-intercepts at 0 and 8 so must have an equation of the form $y = Ax(8 - x)$.

The given measurements and the symmetry of the parabola mean that (4, 5) is a point on the parabola, and so $x = 4$ and $y = 5$ satisfy the equation. This enables us to calculate A:

$$y = Ax(8 - x)$$
$$5 = A \times 4 \times (8 - 4)$$
$$16A = 5$$
$$A = \frac{5}{16} = 0.3125.$$

The area of the cross-section is the area under the curve between $x = 0$ and $x = 8$, so will be given by the integral $\int_0^8 \frac{5}{16} x\,(8 - x)\,\mathrm{d}x$.

$$\int_0^8 \frac{5}{16} x\,(8 - x)\,\mathrm{d}x = \int_0^8 \left(\frac{5x}{2} - \frac{5x^2}{16}\right)\mathrm{d}x$$
$$= \left[\frac{5x^2}{4} - \frac{5x^3}{48}\right]_0^8$$
$$= \frac{5}{2} \times 64 - \frac{5}{48} \times 512 - (0 - 0)$$
$$= 80 - 53\frac{1}{3}$$
$$= 26\frac{2}{3}\,\mathrm{m}^2$$

The volume of material removed can be estimated from $26\frac{2}{3} \times 200 = 5333\,\mathrm{m}^3$.

This is probably an underestimate because the cross-section of the finished tunnel has a smooth surface, but extra material would have been excavated behind the finished surface. Also, the entrance and exit were probably sloping rather than vertical, so some extra material may have been removed outside the location of the finished tunnel.

22.09 Linking algebra, geometry and calculus

We finish this chapter with an example that involves finding points of intersection first. This example links algebra, geometry and calculus.

Example

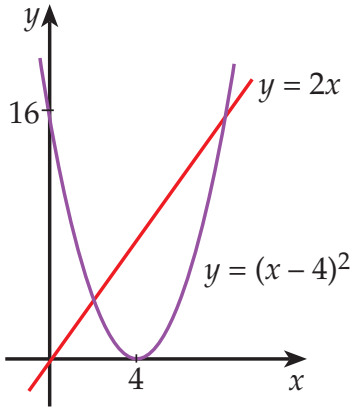
Draw the graphs of the line $y = 2x$ and the parabola given by $y = (x - 4)^2$ on the same diagram. Calculate the area enclosed by the line and the parabola.

Answer

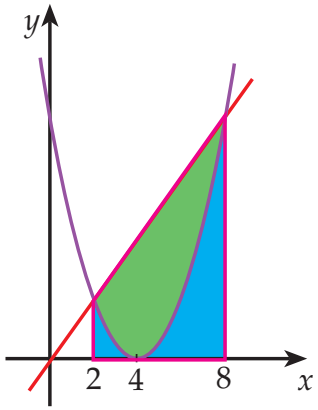
To find the area between the line and the parabola, we first need to find the x-values of the points of intersection.

We solve the equations $y = 2x$ and $y = (x - 4)^2$ simultaneously:

$$(x - 4)^2 = 2x$$
$$x^2 - 8x + 16 = 2x$$
$$x^2 - 10x + 16 = 0$$
$$(x - 2)(x - 8) = 0$$
$$x = 2 \text{ or } 8$$



The required area (coloured green in the diagram) is the difference between the area under the line between $x = 2$ and $x = 8$ (a trapezium, outlined in red), and the area under the parabola (coloured blue).



$$\begin{aligned}
\text{Area} &= \int_2^8 2x \, dx - \int_2^8 (x^2 - 8x + 16) \, dx \\
&= [x^2]_2^8 - \left[\frac{x^3}{3} - 4x^2 + 16x\right]_2^8 \\
&= (64 - 4) - \left[\left(\frac{512}{3} - 256 + 128\right) - \left(\frac{8}{3} - 16 + 32\right)\right] \\
&= (64 - 4) - \left[\left(\frac{512}{3} - 128\right) - \left(\frac{8}{3} + 16\right)\right] \\
&= 60 - \left(\frac{512}{3} - \frac{8}{3} - 128 - 16\right) \\
&= 60 - (24) \\
&= 36 \text{ square units}
\end{aligned}$$

23.01 Rates of change

The gradient of a line is a measure of how quickly y is changing in relation to x. Thus, $\frac{dy}{dx}$ is the rate of change of y with respect to x.

This idea can be extended to other variables. We can use calculus to find rates of change when any two variables are linked by a formula.

We always write rates of change using the $\frac{dy}{dx}$ notation. This makes it easy to identify the two variables involved. For example, if A represents the area of a circle and r represents the radius, then the rate of change of the area with respect to the radius is written as $\frac{dA}{dr}$.

Using the $\frac{dy}{dx}$ notation has a bonus: it also shows how to work out the required rate of change – that is, by differentiating the variable on the top line with respect to the variable on the bottom line.

For example, to find the rate of change of the circumference of a circle, C, with respect to its radius, r, we would use the notation $\frac{dC}{dr}$ and differentiate the formula $C = 2\pi r$ with respect to r.

Example

The area of a circle is given by the formula $A = \pi r^2$. Calculate the rate at which the area is changing with respect to the radius when the radius of the circle is 12 cm.

Answer

The rate of change of the area with respect to the radius is $\frac{dA}{dr}$.

$$A = \pi r^2$$

$$\frac{dA}{dr} = \pi \times 2r = 2\pi r$$

The value of $\frac{dA}{dr}$ when $r = 12$ is $2 \times \pi \times 12 = 24\pi$

$= 75.4$ (3 sf)

23.02 Kinematics – differentiation

- **Velocity** is the rate of change of distance with respect to time.
- **Acceleration** is the rate of change of velocity with respect to time.

We can express velocity as the rate of change of distance, s, with respect to time, t:

$$v = \frac{\mathrm{d}s}{\mathrm{d}t}.$$

Similarly, acceleration, a, is the rate at which velocity changes with respect to time:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}.$$

We give the name **kinematics** to the study of motion and, in particular, to the links between:

s = distance	measured in m
v = velocity	measured in m/s (Sometimes, m/s is written as m s^{-1}.)
a = acceleration	measured in m/s^2 (m/s^2 can also be written as m s^{-2}.)
t = time	measured in s.

The units used in kinematics can vary depending on the context.

- Sometimes, kilometres (km) are used for distance instead of m.
- Time is usually measured in seconds but sometimes it's measured in minutes or hours.

To convert from m/s to km/h and vice versa, use these rules:

1 m/s = 3.6 km/h and 1 km/h $= \frac{5}{18}$ m/s.

KEY POINTS

Common conventions in kinematics

The origin

- All motion is considered to start from an arbitrary point called the **origin**. Motion usually takes place in a horizontal line. If motion is vertical, the origin is taken as ground level.
- The term 'initially' means when the time $t = 0$. This is when motion usually starts.
- Distances above or to the right of the origin are taken as positive. Distances below or to the left of the origin are taken as negative. An object is at the origin when the distance, s, $= 0$.

Velocity

- Positive velocity indicates an object is moving right, or upwards, or in general **away from the origin**.
- Negative velocity indicates an object is moving left, or downwards, or in general **towards the origin**.
- If the velocity, v, $= 0$ the object is **stationary**, or 'at rest'.

Acceleration

- Positive acceleration indicates that an object is **speeding up**.
- Negative acceleration indicates that an object is **slowing down**.
- If the acceleration, a, $= 0$, the object is travelling at **constant speed**.

Note: s, v and a are all functions of t, time:

$$s \xrightarrow{\text{Differentiate}} v \xrightarrow{\text{Differentiate}} a$$

$$a \xrightarrow{\text{Integrate}} v \xrightarrow{\text{Integrate}} s$$

Example

The distance, s, in metres, travelled by a particle is given by the formula $s = 80t - 12t^2$.

a Find an expression for the velocity, and hence find the initial velocity.

b When is the particle at rest?

c Find the acceleration.

Answer

a $v = \dfrac{ds}{dt} = 80 - 24t$

The initial velocity $v(0) = 80 - 24(0) = 80$ m/s.

b The particle is at rest when $v = 0$.

$$v = 0$$

$$80 - 24t = 0$$

$$t = \frac{80}{24} = 3.\dot{3} \text{ seconds}$$

c $a = \dfrac{dv}{dt} = {}^{-}24 \text{ m/s}^2$

Example

A mechanical rabbit starts from the origin. It moves in a straight line, initially moving east.
Its distance from the origin after t seconds is given by $s = (6t - 3t^2)$ m. Which of the following is *false*?

(A) The rabbit slows down in the first second.

(B) The initial velocity of the rabbit is 6 m s^{-1} east.

(C) The velocity of the rabbit at $t = 1$ second is 3 m s^{-1} east.

(D) The acceleration of the rabbit is 6 m s^{-2} west.

Answer

(A) $v = \frac{ds}{dt} = 6 - 6t \Rightarrow a = \frac{dv}{dt} = {}^{-}6$.

Negative acceleration means the rabbit is slowing down. The statement is true.

(B) $v(0) = 6 - 6 \times 0 = 6$

Statement is true: 6 is positive, and means movement east.

(C) $v(1) = 6 - 6(1) = 0$

Statement is false.

(D) $a = {}^{-}6$. The negative sign shows acceleration west rather than east.

Statement is true.

Answer is (C).

23.03 Kinematics – anti-differentiation

Example

A rock is thrown vertically upwards with an initial velocity of 40 m/s from a point 5 m above ground level. The velocity of the rock after t seconds is given by the formula $v = (40 - 10t)$ m/s.

a Find an expression for the height of the rock above ground level at time t.

b Find the maximum height reached by the rock.

c Calculate the distance travelled in the third second.

d When does the rock return to the ground?

Answer

a $s = \int v \, dt$

$= \int (40 - 10t) \, dt$

$= 40t - 5t^2 + c$

When $t = 0$, $s = 5$ (initial height is 5).

$5 = 40 \times 0 - 5 \times (0)^2 + c$

Therefore $c = 5$, i.e. height $s = 40t - 5t^2 + 5$.

b Maximum height is when the velocity, i.e. $\frac{ds}{dt}$, $= 0$.

$40 - 10t = 0$

$t = 4$

Maximum height $= s(4) = 40 \times 4 - 5(4)^2 + 5$

$= 160 - 80 + 5$

$= 85$ m.

c The third second represents all the time between $t = 2$ and $t = 3$. The distance travelled in this time is the height after 3 seconds minus the height after 2 seconds:

$s(3) - s(2) = (40 \times 3 - 5(3)^2 + 5) - (40 \times 2 - 5(2)^2 + 5)$

$= (120 - 45 + 5) - (80 - 20 + 5)$

$= 80 - 65$

$= 15$ m.

Alternatively, the distance travelled in the third second can be

d The rock returns to the ground when the height (s) is 0.

$$40t - 5t^2 + 5 = 0$$

$$t^2 - 8t - 1 = 0 \quad \text{(on division by } {}^{-}5\text{)}$$

$$t = \frac{8 \pm \sqrt{8^2 - 4 \times 1 \times {}^{-}1}}{2 \times 1} \text{ (quadratic formula)}$$

$$= 8.1 \text{ or } {}^{-}0.1 \text{ seconds}$$

Ignore the negative answer: this is a time before the motion started. So, the rock returns to the ground after 8.1 seconds.

23.05 Maxima and minima – enclosed areas

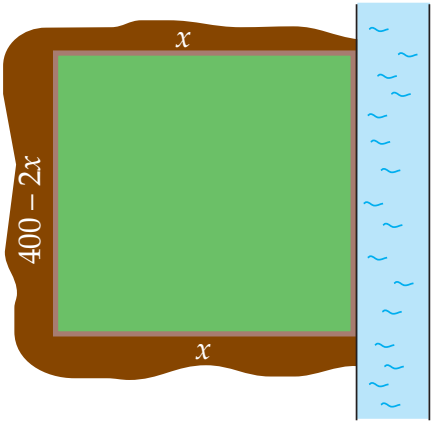
Calculus can be used to find the **maximum** or **minimum values** of quantities, such as area, volume, cost, etc.

The method involves expressing the quantity, for example area, as a function of one variable, and then differentiating to find when the quantity is a maximum or minimum.

You should always check to see whether you have found a maximum or minimum value.

Example

A farmer has 400 m of fencing material available to make a rectangular enclosure. One side of the field is alongside a straight river and does not need to be fenced. Determine the maximum possible area of the field.



Answer

Let one side of the field be x, as shown in the diagram. Then the opposite side is also x, and the third side, parallel to the river, is $400 - 2x$.

$$\begin{aligned}\text{Area} = A(x) &= \text{base} \times \text{height} \\ &= x(400 - 2x) \\ &= 400x - 2x^2\end{aligned}$$

$A'(x) = 400 - 4x = 0$ for a stationary point

$$\begin{aligned}400 &= 4x \\ x &= 100\end{aligned}$$

Check that this is in fact a maximum: the graph of $y = 400x - 2x^2$ would be an upside-down parabola, where the vertex gives a maximum point rather than a minimum.

Thus, the dimensions of the field are 100 m, 200 m and 100 m.

The maximum possible area is $100 \times 200 = 20\,000$ m^2.

23.06 Other applications of maxima and minima

Example

A textile manufacturer makes blankets, and sells them for $25 each. The cost, in dollars, of making x blankets is given by the function: cost $= C(x) = 5000 + 4x + 0.01x^2$.

a What is the cost of producing:

i 100 blankets? **ii** 0 blankets?

b What is the formula for the revenue (money received) from selling x blankets?

c Find the formula for the profit made by selling x blankets. (Hint: profit = revenue – cost.)

d Hence find the number of blankets that the manufacturer should produce in order to maximise profit.

Answer

a i $C(100) = 5000 + 4 \times 100 + 0.01 \times 100^2$

$= 5000 + 400 + 100 = \$5500$

ii $C(0) = \$5000$

Note that \$5000 is the amount of fixed costs that occur, regardless of whether or not any blankets are produced.

b One blanket sells for \$25. x blankets sell for $\$25x$.

c Profit = revenue – cost

$= 25x - (5000 + 4x + 0.01x^2)$

$= 25x - 5000 - 4x - 0.01x^2$

$= {}^{-}5000 + 21x - 0.01x^2$

d Differentiating the profit gives:

$\text{Profit}'(x) = 21 - 0.02x = 0$ for a stationary point

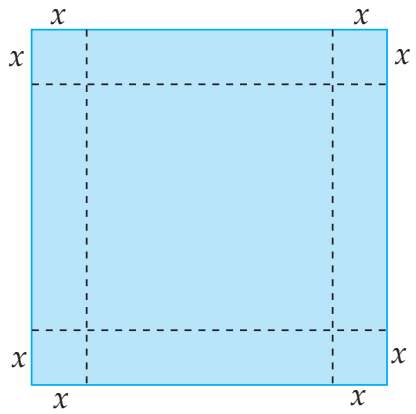
$$21 = 0.02x$$

$$x = \frac{21}{0.02} = 1050$$

Note that $x = 1050$ gives a maximum, because the graph of $y = {}^{-}5000 + 21x - 0.01x^2$ is an upside-down parabola.

Example

A box with an open top is made from a square sheet of cardboard with sides 12 cm, by cutting a square of side x from each corner, and folding along the dashed lines as shown in the diagram.



a Express L, the length of the box, in terms of x.

b Find an expression for V, the volume of the box, in terms of x.

c Differentiate to find $\frac{dV}{dx}$.

d Calculate the maximum volume of the box.

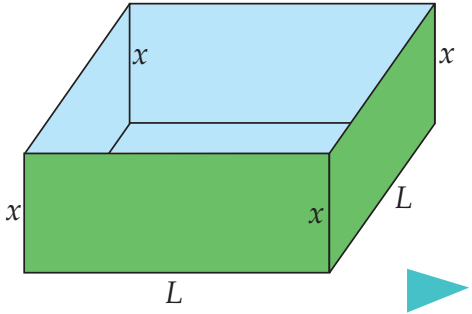
Answer

a $L = 12 - 2x$

b $V = L \times L \times x = xL^2 = x(12 - 2x)^2$

$= x(144 - 48x + 4x^2)$

$= 144x - 48x^2 + 4x^3$



c $\frac{dV}{dx} = 144 - 96x + 12x^2$

d Solve $V'(x) = 0$

$144 - 96x + 12x^2 = 0$

$12 - 8x + x^2 = 0$ or $x^2 - 8x + 12 = 0$

$(x - 6)(x - 2) = 0$

$x = 6$ or $x = 2$

Evaluate the volume for each of these x-values:

$V(2) = 2 \times 8^2 = 128 \text{ cm}^3$

$V(6) = 6 \times 0^2 = 0 \text{ cm}^3$.

Clearly, the maximum volume is 128 cm^3, obtained when $x = 2$ cm.

24.01 Equally likely outcomes

Much of probability is based on the assumption that, in many situations, events are **equally likely to occur** if random choices are made.

- A coin is equally likely to fall either 'heads' or 'tails' up.
- A six-sided die is equally likely to give any number from the set {1, 2, 3, 4, 5, 6}.
- The first ball from the weekly Lotto draw is equally likely to produce any number from 1 to 40 inclusive.

In these situations, we claim that the coin is 'fair' or that the die is 'unbiased'. This is another way of saying that each of the possible outcomes is equally likely.

When outcomes are **equally likely**, the probability of an event is:

$$\frac{\text{number of outcomes favourable for the event}}{\text{total number of outcomes}}.$$

Example

In Lotto, what is the probability of obtaining a number between 23 and 27 inclusive, when the first ball from the 40 possible balls (numbered from 1 to 40) is released?

Answer

Favourable numbers are 23, 24, 25, 26, 27 (five outcomes).

Total number of outcomes is 40.

Required probability is $\frac{5}{40}$ or $\frac{1}{8}$.

Example

A letter is chosen at random from the word ‘mathematics’.
Find the probability that a vowel (a, e, i, o, u) is chosen.

Answer

There are 11 letters, each of which is equally likely. Four of the letters are vowels (a, e and i).
Note that there are two ‘a’s – each of these has to be counted.
The required probability is $\frac{4}{11}$.

Example

A fruit bowl contains four Golden Delicious and five Granny Smith apples (both types have green skin), and seven Gala apples (with red skin). Find the probability that, when an apple is picked at random, it has a green skin.

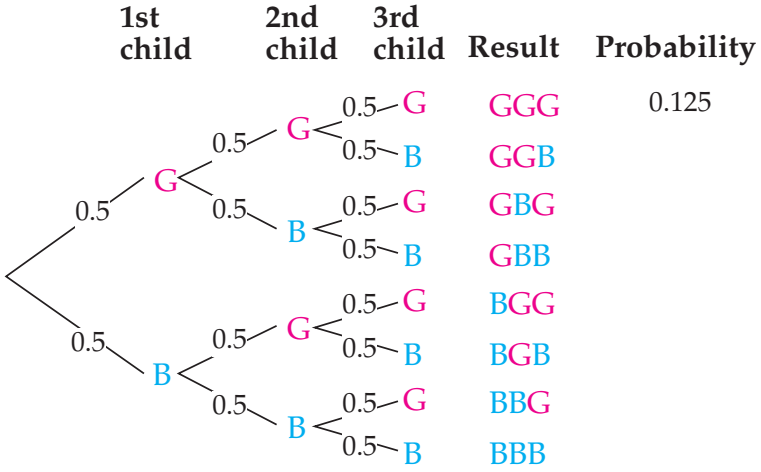
Answer

Nine apples have green skins.
There are 16 apples altogether.
The probability is $\frac{9}{16}$.

24.02 Probability trees

A **probability tree** is a branching diagram used to show alternative outcomes. At each stage in the diagram, there is a choice, dependent on chance.

Here is a probability tree for the different outcomes of a couple having three children. The first two branches represent the two possible outcomes for the first child: girl or boy. Each branch then itself branches into two. These second branches show the two possible outcomes for the second child: girl or boy.

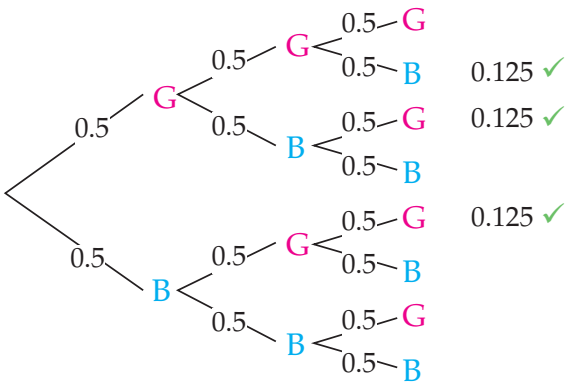


The four branches then each split into two, representing the two possible outcomes for the third child. There are, then, eight possible endings on the diagram: GGG, GGB, etc.

If we assume that, at any stage, boys and girls are equally likely, we can put the probabilities on the tree. Each branch has 0.5 on it. We multiply along the branches to obtain the probability of the outcome at the end. Here the probability of GGG is:

$0.5 \times 0.5 \times 0.5 = 0.125$

To find the probability of a three-child family having exactly two girls, we first mark the favourable branch endings with a ✓ and then add the probabilities to get $0.125 + 0.125 + 0.125 = 0.375$.



KEY POINTS

Adding or multiplying probabilities

You need to be very clear as to when probabilities are **added** and when they are **multiplied**.

- **Multiply** probabilities along branches to get the probability at the end. This is because you want the probability of, say, three girls, which requires the first **and** the second **and** the third to be girls.
- **Add** probabilities when combining different outcomes that are contributing to the desired result. On a probability tree, do this when you have several probabilities at the *ends* of the branches.

To calculate the probability of two girls, we needed to add above, because the outcomes that gave us two girls were GGB **or** GBG **or** BGG.

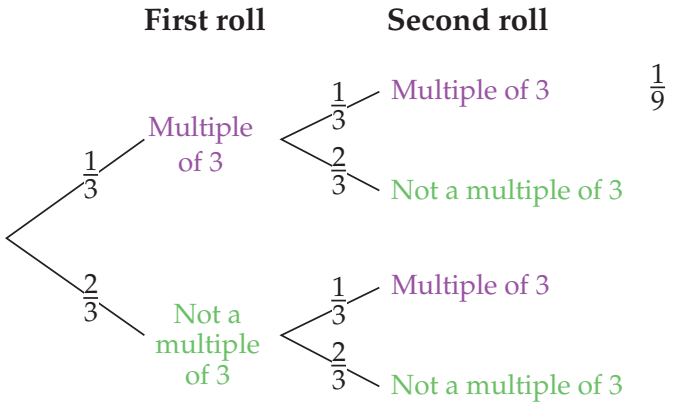
Summary

Action	Where?	Key word
Multiply	Sideways along branches	'and'
Add	Vertically at ends of branches	'or'

Example

Draw the probability tree for when a fair six-sided die is rolled twice, and the required event is that we get a multiple of 3 on both rolls.

Answer



Note that the probability of getting a multiple of 3 on a *single* roll is $\frac{1}{3}$.

This is because two out of the six possible outcomes (i.e. {3, 6}) give a multiple of 3.

The probability of getting a multiple of 3 on both rolls is $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$. We multiply because we want a multiple of 3 on the first roll *and* the second roll.

Example

A plant nursery specialising in selling native trees has a 'money-back' guarantee on two species: kauri and rimu. 40% of the trees sold under guarantee are kauri, and 60% are rimu. The nursery finds that 10% of the kauri trees are returned for a refund, but only 5% of the rimu trees are returned.

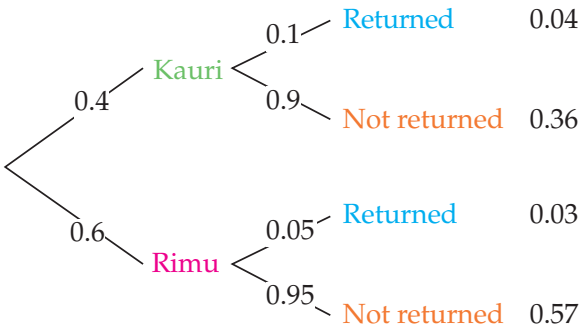
a Draw the probability tree for this question.

b If a tree is chosen at random, calculate the probability that it is a kauri tree and is returned.

c Calculate the probability that a tree chosen at random is returned.

Answer

a



b Multiply the probabilities along the 'kauri' branch and the 'returned' branch to get $0.4 \times 0.1 = 0.04$.

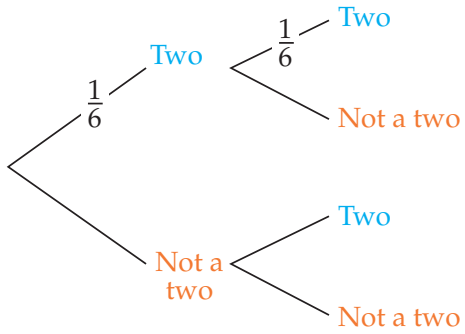
c There are two possible ways in which a tree could be returned. It could be kauri and returned, or rimu and returned. Add the probabilities of these two outcomes to get $0.04 + 0.03 = 0.07$.

24.03 Sampling without replacement

All of the situations so far have assumed that the probability of an event stays the same along the branches.

Example

If a die is tossed twice, the probability of a 2 will be the same on the second branch as it is on the first branch. The die has no memory: a 2 is just as likely on one toss as it is on any other. There is an infinite store of 2s – all of which are just as likely on any toss.



However, as soon as we sample from a **finite** or **fixed number of outcomes**, the probabilities change from branch to branch.

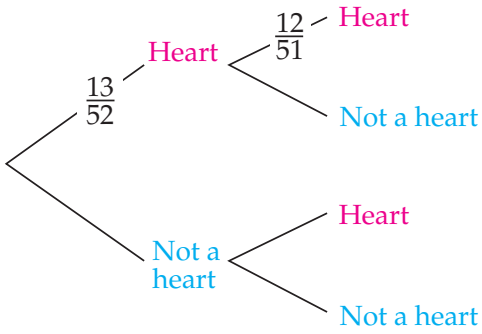
Example

Consider the problem of calculating the probability of getting two hearts from a pack of 52 cards when two cards are drawn in succession. (Note that in a pack of 52 cards, there are 13 hearts.)

- For the first card, the probability that it is a heart is obviously $\frac{13}{52}$, i.e. $\frac{1}{4}$.
- Once the first card has been dealt, there are only 51 cards remaining in the pack. Twelve of these are hearts, because one of the 13 original hearts has already been removed. Therefore, the probability that the second card is a heart is $\frac{12}{51}$.

Following the branches through, the probability that *both* the cards are hearts is:

$$\frac{13}{52} \times \frac{12}{51} = \frac{1}{4} \times \frac{4}{17}$$
$$= \frac{1}{17}$$



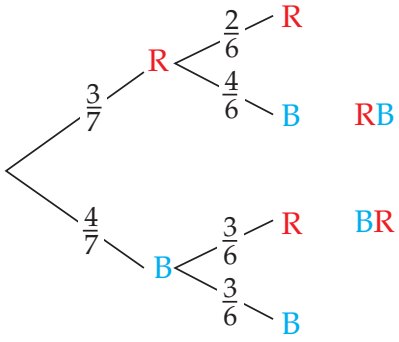
This kind of problem is called **sampling without replacement**, because the card drawn first is *not put back* into the pack of cards before the second card is chosen. If the first card were put back, the probability of getting a heart would remain fixed at $\frac{1}{4}$. But because one of the hearts has been removed, it makes it less likely than $\frac{1}{4}$ that the next card will be a heart.

Example

Two marbles are drawn in succession from a bag that contains three red marbles and four blue marbles. Calculate the probability that one of each colour marble is drawn.

Answer

On the first branches, $P(\text{R}) = \frac{3}{7}$ and $P(\text{B}) = \frac{4}{7}$, obviously.



For the second branches, note that there are only six marbles left in the bag, because one of the original seven has been removed.

- The probability that the second marble is red is $\frac{2}{6}$ if the first one was red (because two of the remaining six marbles are red if one of the original red marbles has been removed).
- The probability that the second marble is red is $\frac{3}{6}$ if the first one was blue (because the original three red marbles are still present).

Likewise, the probabilities for the second marble being blue are influenced by the colour of the first marble.

- If the first is red, four of the remaining six are blue.
- If the first is blue, only three of the remaining six are blue.

The resulting probability tree is shown above.

To have one marble of each colour, we want outcomes RB and BR.

The probability is given by:

$$\frac{3}{7} \times \frac{4}{6} + \frac{4}{7} \times \frac{3}{6} = \frac{12}{42} + \frac{12}{42}$$
$$= \frac{24}{42}$$
$$= \frac{4}{7}$$

In probability situations, drawing two marbles out of a bag together is regarded as the same as drawing one after the other.

24.04 Venn diagrams

In some situations, you cannot multiply probabilities. This is because the events concerned are not independent.

Example

If 20% of New Zealanders have incomes over $70 000 and 30% of New Zealanders travel overseas each year, we cannot draw the conclusion that $20\% \times 30\% = 6\%$ fit into both categories.

The probability that a New Zealander has an income over $70 000 *and* travels overseas would be higher than 6%, because the high income means the New Zealander can afford overseas travel. More information would be needed to find the probability of being in both categories.

This kind of probability is best dealt with by using **sets**. The group of people who fit into both categories is represented by the intersection of two sets. **Venn diagrams** show this clearly.

- The group of all possibilities or people is represented by a large rectangle.
- Smaller groups or sets of people are represented by circles inside the rectangle.
- Overlapping circles show that the smaller groups have items or people in common.

It helps to be familiar with set notation here.

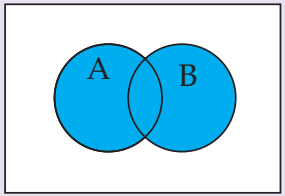
Set notation

KEY POINTS

The **union** of two sets is the set containing the objects or items in *one*, or the *other*, or *both*.

The symbol for union is $\cup$.

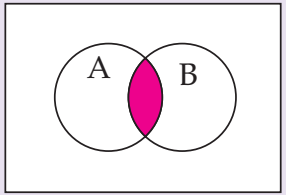
$\mathbf{A \cup B}$



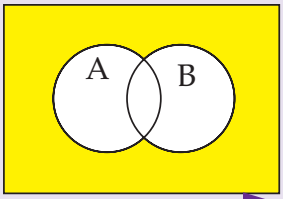
The **intersection** of two sets of the set containing the objects or items in *both only*.

The symbol for intersection is $\cap$.

$\mathbf{A \cap B}$



The set of items or objects that are in neither set is represented by the area outside the circles.

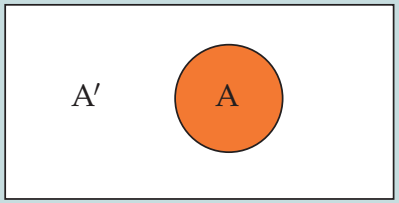


General principles

There are some general principles that apply to all probabilities in Venn diagrams.

The total probability is always 1

A′ is the **complementary** event to A. A′ means the event 'not in A′.



$P(A)$ is the probability inside the circle.
$P(A')$ is the probability outside the circle.

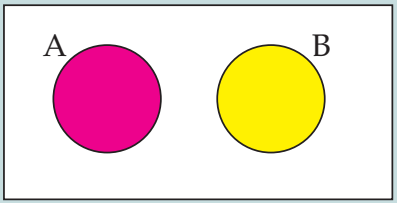
$$P(A) + P(A') = 1$$

Example

The probability that it rains tomorrow is 0.4. The probability that it *does not* rain tomorrow is 0.6.

Mutually exclusive events have nothing in common

For mutually exclusive events, it is not possible to be in both groups at the same time. This is shown on a Venn diagram by two sets that do not overlap. Such sets are described as **disjoint**. This means that their intersection is the empty set { } or Ø.



$A \cap B = \varnothing$

To find the probability of A or B, we add the individual probabilities of A and B:

$$P(A \cup B) = P(A) + P(B)$$

Example

There are 30 sweets in a bag, five of which have a red wrapper and six of which have an orange wrapper. Calculate the probability that, when a sweet is chosen at random from a bag, it has a red or an orange wrapper.

Answer

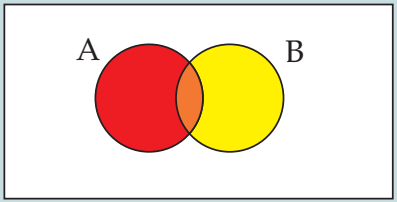
$P(\text{red}) = \frac{5}{30}$ $P(\text{orange}) = \frac{6}{30}$

$P(\text{red or orange}) = \frac{5}{30} + \frac{6}{30} = \frac{11}{30}$

Intersecting events overlap in the Venn diagram

The probability of the intersecting part $P(A \cap B)$ is not usually just $P(A) \times P(B)$. In most situations, $P(A \cap B)$ will be supplied as a piece of extra information.

$A \cup B$ is the **entire area** inside both circles, including the intersection.



To find the probability of $A \cup B$, add the probabilities of A and B. This counts the intersection twice, so subtract the probability of the intersection.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example

$P(A) = 0.4$, $P(B) = 0.2$, $P(A \cap B) = 0.1$.

Calculate $P(A \cup B)$.

Answer

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B). \\ &= 0.4 + 0.2 - 0.1 \\ &= 0.5 \end{aligned}$$

Example

A survey in a developing country has shown that 45% of the homes have refrigerators, 65% have washing machines, and 30% have both. Find the probability that a home chosen at random has:

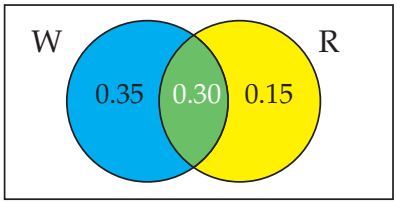
a at least one of these appliances

b neither of these appliances.

Answer

Draw a Venn diagram (see right).

- W = set of houses with washing machines.
- R = set of houses with refrigerators.
- $W \cap R$ = set of houses with both.



Note that the total probability for W is 0.65, so the part of the circle that does not overlap with R has a probability of only 0.35. Similarly for R.

a $P(\text{at least one appliance}) = P(\text{W} \cup \text{R})$

$= 0.35 + 0.15 + 0.30$

$= 0.80$

b $P(\text{neither appliance})$ is given by the region outside both of the circles. This is 0.20, because the total probability is 1.

25.01 Contingency tables

A **contingency table** or a **two-way frequency table** represents the probabilities or the frequencies of events involving two or more variables, which may or may not be independent. The categories of one variable are listed along the top and the categories of the other variable are listed down the side. A two-way frequency table gives the same information as a Venn diagram.

Example

Here is data from a sample of 400 learner drivers when they sat a driver licence test.

If a learner driver is chosen at random, what is the probability that:

a the driver passed?

b the driver was under 18 and failed?

	Passed	Failed	Total
Under 18	119	97	216
Over 18	154	30	184
Total	273	127	400

Answer

a A total of 273 drivers passed, out of 400 altogether.

$P(\text{driver passed}) = \frac{273}{400} = 0.6825.$

b A total of 97 drivers were under 18 *and* failed from a total of 400.

$P(\text{driver was under 18 and failed}) = \frac{97}{400} = 0.2425.$

These calculations show that frequency data can also be presented as probabilities. Calculate these probabilities by dividing each frequency by the overall total, i.e. in this example, divide by 400:

	Passed	Failed	Total
Under 18	0.2975	0.2425	0.54
Over 18	0.385	0.075	0.46
Total	0.6825	0.3175	1.0000

25.02 Determining conditional probability from a contingency table

Conditional probability is the probability of an event occurring given that another event also occurs. Extra information is provided, or assumptions are made, that influence the occurrence of the event.

For example, when drawing cards one after the other without replacement from a pack of 52 well-shuffled playing cards, if you knew the first two cards were red, this information tells us that it is *more likely* that the third card would be black. Why?

In a contingency table, calculating a conditional probability usually involves limiting the possibilities to one row or column only.

A key term that indicates conditional probability is involved is 'given that'. This term indicates that, instead of calculating probability over *all* possibilities (the entire sample space), the possibilities are *limited to a subset* only – usually one row or column of the table.

Example

This table gives information about simplified weather forecasts over a 90-day period in a region of New Zealand.

		Forecast		
		Dry	Wet	Total
Actual weather	Dry	49	8	57
	Wet	5	28	33
	Total	54	36	90

Calculate the probability that it actually rains, *given that* it was forecast to rain.

Answer

The key information is that the *forecast of rain was given*, so we calculate the probability out of only the 36 days that rain was forecast:

Of those 36 days, it rained on 28.

The required probability is $\frac{28}{36} = \frac{7}{9} = 0.78$ (2 dp).

25.03 Risk and relative risk

Risk can be thought of as probability. The term 'risk' is usually used in connection with events that have unpleasant consequences (e.g. death, serious disease, etc.) and, although some of the events may be subject to uncertainty, sometimes there are factors relating to exposure or otherwise that can either reduce the risk or make it worse.

For example, the risk of injury or death from a serious road crash is reduced for someone who wears a safety-belt, and is increased for those who drive after drinking alcohol or who use a cell phone while driving. Whether a crash occurs or not is still uncertain (subject to some randomness or unpredictability) but some of the factors associated with driving can either increase or decrease the probability that the crash occurs.

Relative risk

Here we consider **relative risk** – that is, the likelihood of the event when given factors are taken into account. The relative risk of an event for a particular group or to a certain kind of exposure describes how likely this risk is compared to the risk of the same event for another group or to a different kind of exposure.

Examples

1. The relative risk of a smoker developing lung cancer compared to a non-smoker developing lung cancer – here the two groups (smokers and non-smokers) are complementary.
2. The relative risk of drowning in a swimming pool compared to drowning in any body of water – here the first group is those who swim in a pool, and this is a subset of the second group, which is those who swim anywhere, including a pool.

How do we calculate relative risk?

Relative risk of an event for group A compared to the risk of the same event for group B is the ratio of the probability of the event for group A to the probability of the event for group B.

Example

This table shows data for a study of deaths from lung cancer versus other causes of death in the United States. The table also shows whether the person was a smoker or not.

(*Source*: http://users.rcn.com/jkimball.ma.ultranet/BiologyPages/E/Epidemiology.html)

	Smoker	Non-smoker
Lung cancer	397	37
Other causes	6919	4614
Total	7316	4651

Risk of a smoker dying of lung cancer: $\frac{397}{7316} = 5.43\%$.

Risk of a non-smoker dying of lung cancer: $\frac{37}{4651} = 0.80\%$.

Based on this study, the relative risk of dying of lung cancer for a smoker compared to a non-smoker dying of lung cancer, is $\frac{5.43}{0.80} = 6.8$.

That is, a smoker is almost seven times more likely to die of lung cancer than a non-smoker.

Relative risk is not a probability. Relative risk can have values greater than 1 (e.g. relative risk of cancer for a smoker compared to cancer for a non-smoker) or less than 1 (e.g. risk of head injury for a cyclist who wears a crash helmet compared to risk of head injury for all cyclists).

25.04 Expected value

When a process involving probability is repeated a large number of times, the results are fairly predictable.

Example

If a fair coin is tossed 1000 times, we should get close to 500 heads. We would not be surprised if we only got 480 heads, though, because the coin-tossing is a random process. But if we obtained 800 heads, there would be something wrong!

In theory, we should get exactly 500 heads in the above example. This concept is called **expected value**. It is a kind of theoretical average, or ideal result, that should occur.

The expected value need not be one of the possible values. For example, the expected number of girls in a three-child family, assuming that boys and girls are equally likely, is 1.5. Of course, it is not possible to have 1.5 girls in a family – but what we are saying is that half of the total number (in this case, half of three, i.e. 1.5) is the theoretical average.

How do we work out the expected value?

When a process, involving individual outcomes with a fixed probability (p), is repeated a fixed number of times (n), the expected value is given by:

$$\text{Expected value} = n \times p \text{ or } np.$$

Example

A greengrocer knows that the probability of individual avocados from a certain orchard being delivered in unripe condition is 0.17. In a shipment of 400 avocados, how many could be expected to be unripe?

Answer

Expected value $= n \times p = 400 \times 0.17 = 68$.

Expected value from probability tables

If we know all the possible values that a variable can take, as well as their probabilities, we can calculate the expected value as follows. We make up a table with the values on top and their probabilities below:

x	…
P(x)	…

The expected value is the sum of (values × probability for each value).

Example

A sheep-farmer knows that the ewes on his property will produce lambs each year as follows.

Number of lambs	0	1	2	3
Probability	0.12	0.69	0.18	0.01

Calculate the expected number of lambs produced per ewe.

Answer

Multiply each of the possible values for the number of lambs by its probability, and add up.

$$\begin{aligned}\text{Expected value} &= 0 \times 0.12 + 1 \times 0.69 + 2 \times 0.18 + 3 \times 0.01 \\ &= 1.08\end{aligned}$$

We interpret this answer by saying that the theoretical average number of lambs born per ewe is 1.08. If the farmer had a large number of ewes, he could multiply the number of ewes by 1.08 to estimate the number of lambs produced per year.

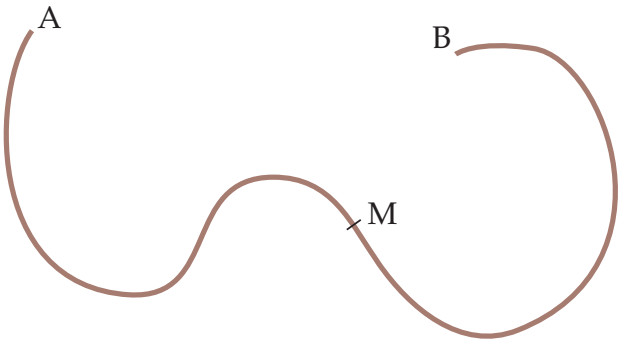
25.05 Continuous probability

How do we calculate the probability that a piece of string is cut closer to one end (A) than the other (B)?

We do this by considering a fraction that compares two lengths:

$$\frac{\text{length in which a point is closer to A than to B}}{\text{total length}} = \frac{AM}{AB} = \frac{1}{2}$$

(where M is the midpoint of the piece of string).



This idea is very similar to the definition of probability for equally likely outcomes, in which we compare:

$$\frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}.$$

The difference is that, for a piece of string, we cannot count the number of possible places where the string can be cut (there is an infinite number). However, we *can* measure the length where the string could be cut, and compare it with the total length.

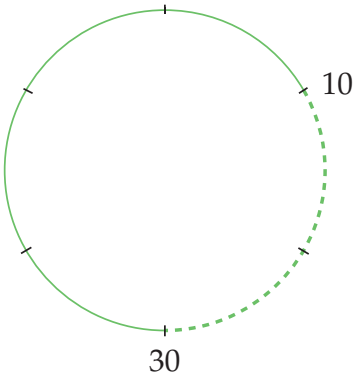
- Probabilities that can only be calculated from measurements are called **continuous**.
- Probabilities that are calculated by counting outcomes are called **discrete**.

Example

A stopwatch is stopped at random. Calculate the probability that the second hand is between 10 and 30 seconds.

Answer

Draw a clock.



Compare the fraction of the circle in which the second hand could be (between 10 and 30) with the whole circle:

$$\frac{20 \text{ seconds}}{60 \text{ seconds}} = \frac{1}{3}$$

An interesting paradox is that continuous measurements can never actually be equal to any particular value. In the example above, we want the second hand to be somewhere between 10 and 30. We do not need to state whether 10 and 30 themselves are included or excluded, because a time will never be exactly 10 seconds or 30 seconds.

To be exactly 10 seconds, the time would have to be 10.000 000 000 000 0 … seconds, which is unmeasurable! All that we can say with certainty is that a time is between two values, rather than equal to one value.

26.02 Standard deviation

The standard deviation (symbol s) of a set of numbers is a measure of their spread about the mean.

Example

This example shows how to calculate the standard deviation of a set of numbers (here we use the numbers 2, 5, 7, 9 and 17).

Answer

1 Calculate the *mean* (symbol $\bar{x}$):

$$\bar{x} = \frac{2+5+7+9+17}{5} = \frac{40}{5}$$

i.e. the mean is 8.

2 Write the *difference of each number from the mean*. This gives:

	$(2-8)$	$(5-8)$	$(7-8)$	$(9-8)$	and	$(17-8)$
i.e.	$^{-}6$	$^{-}3$	$^{-}1$	1	and	9

We wish to 'average' out these differences in some way. However, if we used them as they are, the differences add to 0 (which is not very helpful). Therefore, we square the difference to make them all positive, remembering that later on we take the square root to undo the squaring.

3 Squaring the differences gives:

$$36 \quad 9 \quad 1 \quad 1 \quad \text{and} \quad 81$$

4 Averaging these squares gives:

$$\frac{36+9+1+1+81}{5} = \frac{128}{5}$$

i.e. 25.6

5 Take the square root to 'undo' the squaring earlier: 5.060

The standard deviation of the numbers {2, 5, 7, 9, 17} is 5.060.

Example

(Multichoice) Which of the following has the greatest standard deviation?

(A) {1, 3, 3, 5} (B) {30, 30, 30, 30}

(C) {7, 7, 7, 7} (D) {2, 6, 8, 12}

Answer

There are two possible approaches. Remember that standard deviation is a measure of spread. This means the set that has the greatest range, i.e. (D), is probably the correct answer. But check by calculating the standard deviation for each set:

(A) 1.414 (B) 0

(C) 0 (D) 3.606

(D) has the greatest standard deviation.

Customers want consistency in service. Courier and postal delivery services know this. A common complaint is: 'All I want to know is how many days ahead I have to post my grandmother's birthday card.'

In statistical terms, the customer wants a lower standard deviation.

Calculating the standard deviation from a frequency table

A simple frequency table lists just the possible values of data (usually headed x) and how often these values occur, which is the frequency, f. To calculate standard deviations, we add three new columns to the frequency table.

New columns

x	f	$x \times f$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
Totals	$\sum f$	$\sum xf$		$\sum f(x-\bar{x})^2$

The mean, $\bar{x}$, is $\dfrac{\sum xf}{\sum f}$.

The standard deviation is $\sqrt{\dfrac{\sum f(x-\bar{x})^2}{\sum f}}$.

Example

John took a survey of the 20 boys in his class to find how many pairs of shoes each boy owned. The frequency distribution of the numbers obtained is given here.

x	f
1	1
2	4
3	9
4	6
Total	**20**

Calculate the standard deviation of the numbers.

Answer

Complete a full frequency table:

x	f	$x \times f$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
1	1	1	4	4
2	4	8	1	4
3	9	27	0	0
4	6	24	1	6
Totals	**20**	**60**		**14**

$$\text{Mean} = \bar{x} = \frac{\sum xf}{\sum f} = \frac{60}{20} = 3$$

$$\text{Standard deviation, } s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}}$$

$$= \sqrt{\frac{14}{20}}$$

$$= \sqrt{0.7}$$

$$= 0.837 \ (3 \text{ sf})$$

See the 'Coffee cups' investigation in this chapter (page 404) to see how you could use a **spreadsheet** to calculate the standard deviation of data presented in a frequency table.

26.03 Normally distributed data – 1 sd, 2 sd and 3 sd

Many sets of data collected in nature and other situations fit what is called a **normal distribution**.

- Most measurements are near the middle.
- There are a few extreme values either below or above the mean which have unusually small or large measurements.

Normally distributed data is described by giving the **mean** (the middle value) and the **standard deviation** (a measure of the spread, introduced above).

Examples

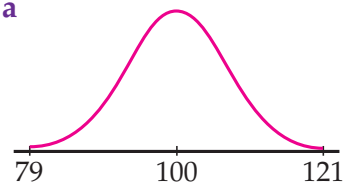
a The length of pine needles is normally distributed, with a mean of 100 mm and standard deviation of 7 mm.

b The weights of suitcases checked in at an airport are normally distributed, with mean
weight 19 kg and standard deviation 5 kg.

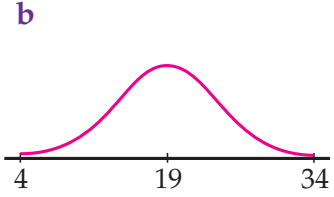
c The lifetime of a light-bulb is normally distributed, with mean 900 hours and standard deviation 150 hours.

The graph for each of the above fits a **bell-shaped curve**, as drawn below.

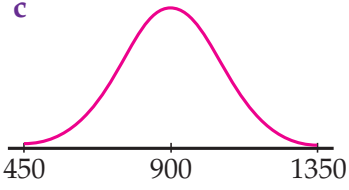
a



b



c



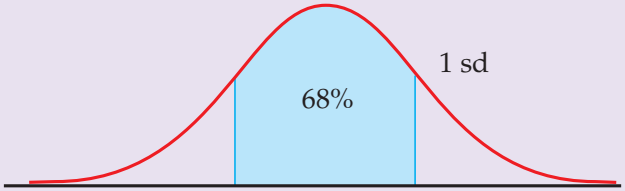
Although each curve has a different central point (mean) and different spread (standard deviation), all three curves share the common shape.

KEY POINTS

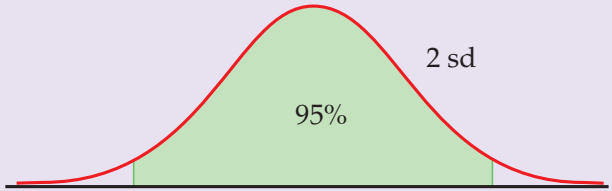
In questions involving normally distributed data, you will usually be given the mean and the **standard deviation** (sd).

Any data that is normally distributed fits certain proportions. These results apply regardless of whether we are considering lengths of leaves, or weights of suitcases, or lifetimes of light-bulbs, etc.

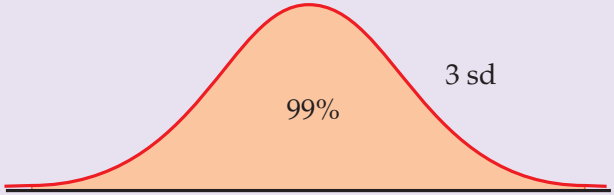
1 **68%** of the data lies within **1 standard deviation** (either side) of the mean. It is **likely** or **probable** that the data will be in this region.



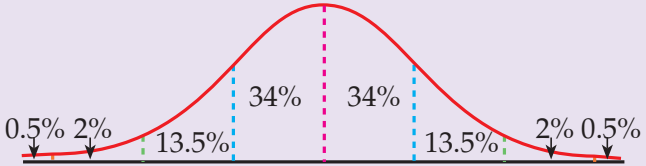
2 **95%** of the data lies within **2 standard deviations** (either side) of the mean. It is **very likely** or **very probable** that the data will be in this region.



3 **99%** of the data lies within **3 standard deviations** (either side) of the mean. It is **almost certain** that the data will be in this region.



4 The normal curve is symmetrical. The probabilities of all possible events add to 100% or 1. These properties allow us to subdivide the area below the curve:



Example

An architect chooses slate for a conservatory floor. Each piece is to measure 300 mm $\times$ 300 mm $\times$ 15 mm minimum thickness. Because slate is a natural product, each piece varies in weight, but the weights are known to be normally distributed with a mean of 945 g and a standard deviation of 10 g.

a What percentage of pieces weigh between 925 g and 965 g?

b What percentage of pieces weigh more than 935 g?

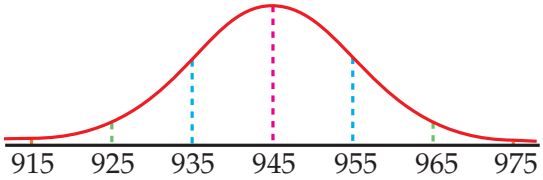
c Below what weight will a piece of slate almost certainly be?

d 400 pieces of slate are ordered for the floor. Estimate how many will weigh less than 925 g.

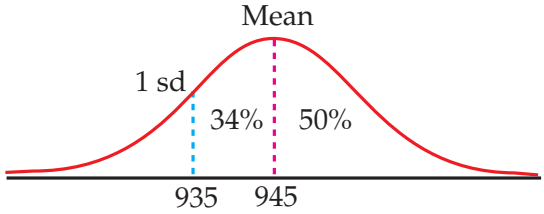
Answer

First draw a normal curve, with a mean of 945 and a standard deviation of 10.

a 925 g and 965 g are each 2 standard deviations below and above the mean, respectively.
This corresponds to a 95% probability.



b 935 g is 1 standard deviation below the mean:

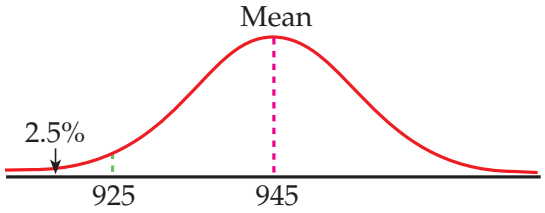


34% (half of 68%) of the weights lie between the mean and 1 standard deviation below the mean; 50% of the weights lie above the mean.
Therefore, 84% weigh more than 935 g.

c The wording 'almost certainly' corresponds to 3 standard deviations. Weights will almost certainly be below a value that is 3 standard deviations above the mean:

$945 + 3 \times 10 = 945 + 30 = 975$ g.

d 925 g is 2 standard deviations below the mean. 2.5% of the weights will be below this point:



Calculate 2.5% of the total of 400 pieces ordered: $2.5\% \times 400 = 10$ pieces.

Notice how the standard deviation gives the information needed to write in the scale on the base of the normal distribution diagram.

26.04 Standard normal tables

Each set of data that is normally distributed has a different mean and a different standard deviation. It is therefore impossible to give probabilities for every situation. Instead, we convert to what is called the standard normal distribution. This has a normal curve with a mean of 0 and a standard deviation of 1.

Special tables give probabilities between zero and any positive value on a standard normal curve. These can be used to give probabilities, between any two values, for data that is normally distributed.

Symbols for mean and standard deviation

An ordinary normal curve (e.g. lengths of leaves, weights of suitcases or lifetimes of light-bulbs) has a mean of μ and standard deviation of σ.

To convert an ordinary normal measurement (x) to a standard normal measurement (z), use the equation:

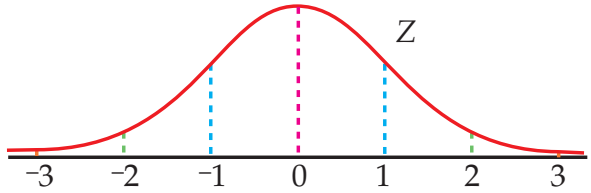
$$z = \frac{x - \mu}{\sigma}.$$

The z-value tells us how many standard deviations x is away from the mean. Almost all normally distributed data is within 3 units of standard deviation of the mean.

The standard normal curve

The scale under the curve is labelled from $^{-}3$ to 3.

- 0 shows the location of the mean.
- $^{-}3$ represents 3 units of standard deviation *below* the mean.
- 3 represents 3 units of standard deviation *above* the mean.



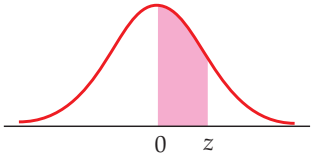
Tables for the standard normal distribution

Here is an extract from standard normal probability tables.

The full standard normal probability tables are provided in pdf format on this *Theta Mathematics* student CD.

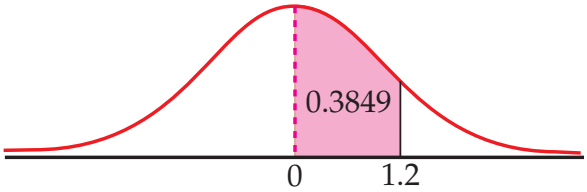
AREAS UNDER THE NORMAL CURVE

This table gives the area under the standard normal curve between 0 and z, shaded in the diagram.



z	0.00	0.01	0.02	0.03	0.04	0.05
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088
0.6	0.2258	0.2291	0.2324	0.2357	0.2389	0.2422
0.7	0.2580	0.2612	0.2642	0.2673	0.2704	0.2734
0.8	0.2881	0.2910	0.2939	0.2967	0.2996	0.3023
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678

These tables give, for example, $P(0 < Z < 1.2) = 0.3849$ (see the graph below).



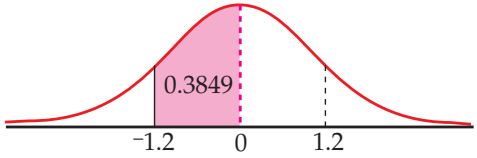
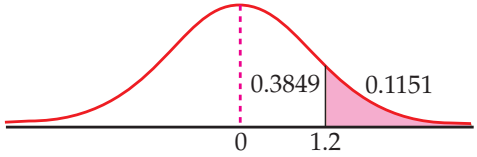
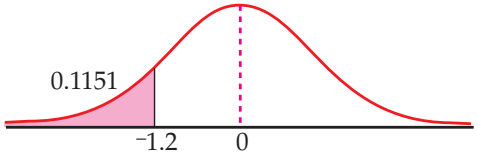
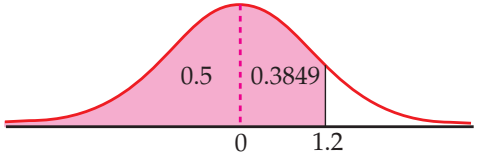
This means that 38.49% of normally distributed data lies between the mean and 1.2 units of standard deviation above the mean.

The standard normal curve is symmetrical.

The total probability above the mean = 0.5 = the total probability below the mean.

We can use the symmetry properties of the curve when calculating areas underneath.

Examples

	Probability	Graph
a	$P(^{-}1.2 < Z < 0) = 0.3849$	
	Probability	**Graph**
b	$P(Z > 1.2) = 0.5 - 0.3849 = 0.1151$	
c	$P(Z < {}^{-}1.2) = 0.1151$	
d	$P(Z < 1.2) = 0.5 + 0.3849 = 0.8849$	

26.05 & 26.06 Converting to the standard normal

The key to working with questions asking for probabilities of normally distributed items is to work with **standard normal probabilities**.

We use x to represent a value from data that is normally distributed. This value is converted to z, which is the corresponding standard normal value. z represents the number of units of standard deviation that x is away from its mean.

Use the equation:

$$z = \frac{x - \mu}{\sigma},$$

where μ is the **mean**, and σ is the **standard deviation**.

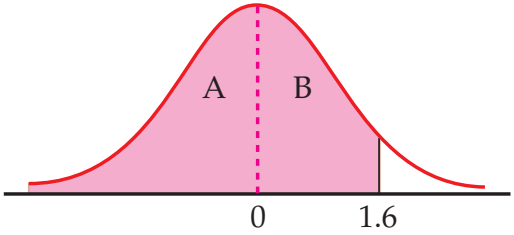
Any particular normal distribution is specified exactly if the mean (μ) and standard deviation (σ) are known. These two numbers are called the parameters of the normal distribution.

Example

X has a normal distribution with a mean of 26 units and a standard deviation of 5 units. Show how to calculate the probability that X has a value less than 34.

Answer

$$P(X < 34) = P\left(Z < \frac{34 - 26}{5}\right)$$
$$= P(Z < 1.6)$$



On the standard normal curve, this probability is represented by all the area to the left of 1.6.

By symmetry, the area of the part labelled A is 0.5, and the area of the part labelled B is in the standard normal tables.

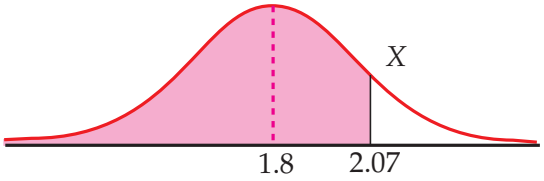
$$P(Z < 1.6) = P(Z < 0) + P(0 < Z < 1.6)$$
$$= 0.5 + 0.4452$$
$$= 0.9452$$

26.07 Applications of the normal distribution

Example

The lengths of Christmas tree branches from a pine-tree plantation can be assumed to be normally distributed, with a mean length of 1.8 m and a standard deviation of 0.2 m. What percentage of branches would measure less than 2.07 m?

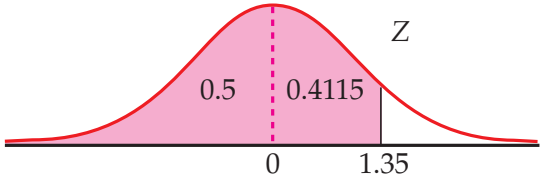
Answer



Draw a normal curve with
$\mu = 1.8$ and $\sigma = 0.2$
to get a feeling for the question.
Calculate $P(X < 2.07)$:

$$P(X < 2.07) = P\left(Z < \frac{2.07 - 1.8}{0.2}\right) \text{ (convert to standard normal)}$$

$$= P(Z < 1.35)$$



$= 0.5 + 0.4115$ (from tables)

$= 0.9115$ (i.e. 91.15%).

26.08 Calculating an expected value

To calculate an expected value in the long run, multiply the total number of trials by the probability.

Example

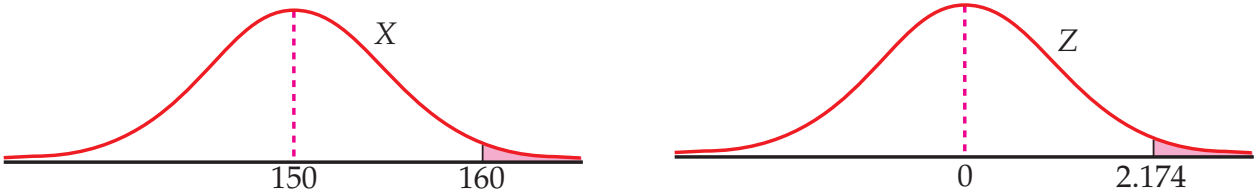
A factory makes glass bottles for filling with lemonade. The weights of the bottles are normally distributed, with a mean of 150 g and a standard deviation of 4.6 g. If a bottle weighs more than 160 g, it is too heavy and must be rejected. From a random sample of 5000 bottles, how many may be expected to be rejected?

Answer

First calculate the probability that a single bottle is overweight:

$$P(X > 160) = P\left(Z > \frac{160 - 150}{4.6}\right) \text{ (convert to standard normal)}$$

$$= P(Z > 2.174)$$



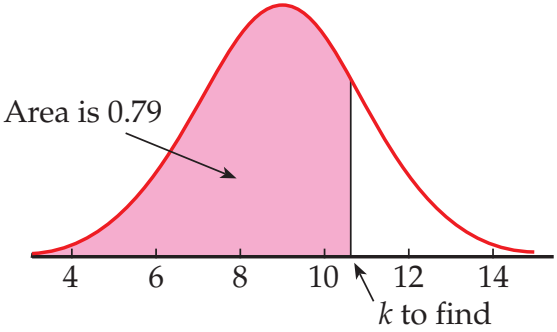
From tables: $P(Z > 2.174) = 0.5 - 0.4852 = 0.0148$

Out of 5000 bottles, 1.48% of them could be expected to be overweight – i.e. approximately 74.

26.09 & 26.10 'Inverse normal' problems

Here we are working 'backwards' from a particular probability to determine a value that gives that probability. This process is described as working out an 'inverse normal' value.

Consider the diagram drawn here:



In general, we know the area under the curve, and want to determine the value on the x-axis that gives that area. Expressed mathematically, the problem is: 'Determine k so that $P(X < k) = 0.79$.'

There are two possible approaches to obtain normal 'inverses':

- use standard normal tables, followed by a conversion to a distribution with the given mean and standard deviation
- calculate the value directly using a calculator, computer or spreadsheet.

Using traditional normal tables to obtain normal 'inverses'

So far, we have calculated probabilities after converting an x-value to a z-value, and then looking up an entry in the normal distribution tables.

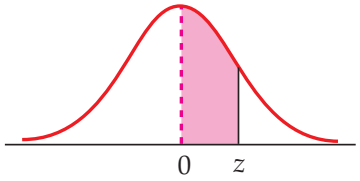
Instead of using a z-value and looking up the probability in the tables, we search for the probability in the main body of the tables first and then read off the z-value from the side (and top, for more accuracy) which gives that probability.

Example

Calculate the value of k for which $P(0 < Z < k) = 0.3770$.

Answer

In this extract from the normal distribution tables, the value 0.3770 is highlighted. By reading off the side and top (the values highlighted in red), we see this corresponds to a z-value of 1.16.



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123
0.6	0.2258	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454
0.7	0.2580	0.2612	0.2642	0.2673	0.2704	0.2734	0.2764
0.8	0.2881	0.2910	0.2939	0.2967	0.2996	0.3023	0.3051
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279

Example

Calculate the value of k for which $P(0 < Z < k) = 0.12$.

Answer

The closest entry in the tables below 0.12 is 0.1179, which corresponds to a z-value of 0.30. This differs from the given 0.12 by 0.0021. The closest value to 0.0021 in the differences column on the right is given by the entry of '22' in the column headed '6'. This '6' gives the next figure of accuracy for 0.30.

i.e. $k = 0.306$.

More generally, a percentage or probability is given and an x-value (for any normal distribution) has to be calculated. We do this by first calculating the standard normal z-value, and then converting to the corresponding x-value using:

$$z = \frac{x - \mu}{\sigma}.$$

A more convenient rearrangement of this formula is:

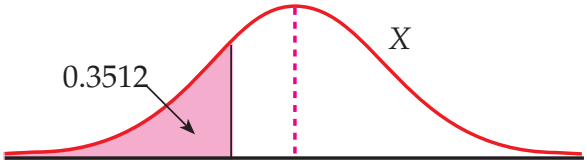
$$x = \sigma z + \mu.$$

Example

X has a normal distribution with a mean of 14 and a standard deviation of 5. $P(X < k) = 0.3512$. Calculate the value of k.

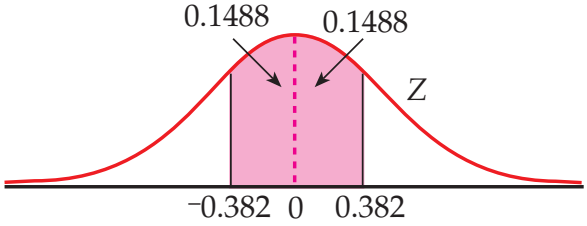
Answer

First, draw a normal curve to get a feeling for the question.



Note that k is below the mean, and will therefore have a negative z-value.

The probability between k and the mean is $0.5 - 0.3512 = 0.1488$.



0.1488 gives a z-value of 0.382 above the mean.

Therefore, by symmetry, the z-value below the mean will be $^{-}0.382$.

Substitute into $x = \sigma z + \mu$.

$k = 5 \times {}^{-}0.382 + 14 = 12.09$

26.11 Applications using 'inverse normal' probabilities

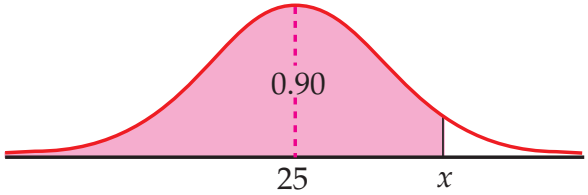
Example

Natalie cycles to her grandmother's house every day after school. The time for the journey is normally distributed, with a mean of 25 minutes and a standard deviation of 4 minutes. There is a 90% probability that Natalie arrives at her grandmother's house before 4 pm. When does she leave school?

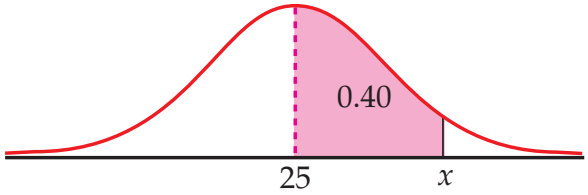
Answer

(Using tables)

First find the time below which 90% of her journeys take. We want the shaded area under the normal curve to be 0.90.



This means the area between 25 and the required time is 0.40.



1.281 units of standard deviation gives a probability of $P(0 < Z < z) = 0.40$.

Substituting into $x = \sigma z + \mu$.

$x = 4 \times 1.281 + 25 = 30.12$ minutes.

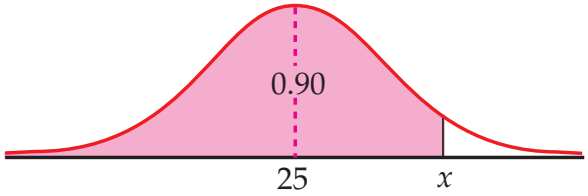
This means 90% of all journeys take under 30.12 minutes.

Thus, Natalie should leave at (4 pm – 30 minutes) = 3:30 pm to have a 90% probability of arriving at her grandmother's house by 4 pm.

Answer

(Using appropriate technology)

Draw a diagram that shows how the time for the journey is distributed to help interpret the information.



Entering NORMINV(0.9,25,4) into Excel™ gives 30.126 208.

This means 90% of all journeys take under 30.13 minutes.

Thus, Natalie should leave at (4 pm – 30 minutes) = 3:30 pm approximately to have a 90% probability of arriving at her grandmother's house by 4 pm.

27.01 Simulation

A **simulation** imitates a real situation and is supposed to give similar results. Hence, a simulation acts as a predictor of what should actually happen. It is a model in which repeated experiments are carried out for the purpose of estimating what might occur in real life. In a simulation, a computer program or spreadsheet formulae can be constructed to follow assumed probabilities.

In this section, we focus on simulating different probability situations. We can allocate random numbers to events so that, in the long run, their relative frequency is supposed to be approximately the theoretical probability.

KEY POINTS

Advantages of simulating probability experiments

- Often there is no cost.
- Because the simulation can be carried out quickly, there is a saving in time and effort.
- Some processes may be complex, and make it difficult to calculate exact probabilities.
- It is easy to keep count of the number of trials.
- If a spreadsheet is used, there is a printed record of what happens.

Disadvantages of simulating probability experiments

- It may be difficult to check you are actually using the right formulae.
- Some knowledge of spreadsheet formulae, or commands and functions on a CAS calculator or computer software can be required.
- Sometimes, there is lack of access to a computer.
- Some probabilities take a long time to 'settle down' to the exact value, or close to it.

Example

Simulating tossing a fair coin

A very simple example of a simulation is where the action of tossing a coin can bemodelled by using random numbers.

First we assume that we can obtain random decimals between 0 and 1. Random decimals can be obtained from a spreadsheet, a scientific calculator or special books of statistical tables. Any decimal is equally likely, so the process matches the randomness of tossing a coin that we require.

We allocate groups of random decimals to each possible outcome.

- Heads: 0.000 000–0.499 999
- Tails: 0.500 000–0.999 999

Here is an example of 10 random decimals generated by a spreadsheet and the allocation of the outcomes 'Heads' and 'Tails'. To **repeat** the experiment, place the cursor in any cell, and press the F9 key. This spreadsheet is called 'Heads or Tails.xls' and is provided on the *Theta Mathematics* student CD and on the www.mathematics.co.nz website.

	A	B
1	**Random decimal**	**Outcome (Tails or Heads)**
2		
3	0.524021877	T
4	0.227032326	H
5	0.895802089	T
6	0.883645423	T
7	0.493027573	H
8	0.235123458	H
9	0.358803458	H
10	0.020092926	H
11	0.543948581	T
12	0.330344435	H

Formulae:

- in column A, each cell has the formula =RAND()
- in column B, the formula in cell B3 is =IF(A3<0.5,'H','T'). This allocates 'H' for random decimals less than 0.5, and 'T' otherwise.

Advantages of this particular simulation compared with coin-tossing:

- the simulation saves finding a coin!
- it avoids the impossible task of checking whether the coin being used is fair
- the 'coin-tossing' can be carried out instantly
- there is no risk that the process is not being carried out correctly, using a wrong coin-tossing technique
- it is no more difficult to toss the coin 1000 times than only 10 times
- the experiment can be repeated over and over.

Example

Estimate the probability that the four members of a family were each born on a different day of the week, by carrying out a simulation.

Answer

We assume that each of the seven days of the week is equally likely – unlikely today when induced births are common.

This assumption means each day of the week has a probability of $\frac{1}{7}$ and we can allocate random numbers to the days of the week as follows.

Day of the week	Random number
Sunday	1
Monday	2
Tuesday	3
Wednesday	4
Thursday	5
Friday	6
Saturday	7

In a spreadsheet, the formula =RANDBETWEEN(1,7) gives random numbers between 1 and 7.

	A	B	C	D	E	F	G	H	I	J
1	Number of trials	Random numbers for first, second, third and fourth persons in family								All different? (1 = yes)
2	1	2	6	3	5	MON	FRI	TUE	THU	1
3	2	4	2	6	5	WED	MON	FRI	THU	1
4	3	4	1	2	6	WED	SUN	MON	FRI	1
5	4	6	1	6	1	FRI	SUN	FRI	SUN	0
6	5	3	1	5	6	TUE	SUN	THU	FRI	1
7	6	2	7	5	6	MON	SAT	THU	FRI	1
8	7	7	1	4	3	SAT	SUN	WED	TUE	1
9	8	3	2	4	2	TUE	MON	WED	MON	0
10	9	3	1	3	3	TUE	SUN	TUE	TUE	0
11	10	6	7	1	3	FRI	SAT	SUN	TUE	1
12	11	3	5	1	6	TUE	THU	SUN	FRI	1
13	12	4	1	4	6	WED	SUN	WED	FRI	0
14	13	5	3	6	5	THU	TUE	FRI	THU	0
15	14	1	6	2	1	SUN	FRI	MON	SUN	0
16	15	3	3	4	2	TUE	TUE	WED	MON	0
17	16	2	4	7	4	MON	WED	SAT	WED	0
18	17	6	4	2	1	FRI	WED	MON	SUN	1
19	18	6	2	6	3	FRI	MON	FRI	TUE	0
20	19	1	7	4	6	SUN	SAT	WED	FRI	1
21	20	2	4	1	7	MON	WED	SUN	SAT	1
22										11
23										
24		Relative frequency where all four are different								
25					0.55					
26										

To simulate a large number of four-person families, we generate quadruples of these random numbers a large number of times and count how many have different days of the week for all four family members. (Note: in the spreadsheet extract here we simulate
20 families; this is not enough to give a very accurate estimate.)

In column A, we place numbers from 1 onwards to keep track of the number of trials. In columns B to E, we show the random number that represents each person's day of birth. (Columns F to I are optional – in these columns, we can show the resulting day of the week.)

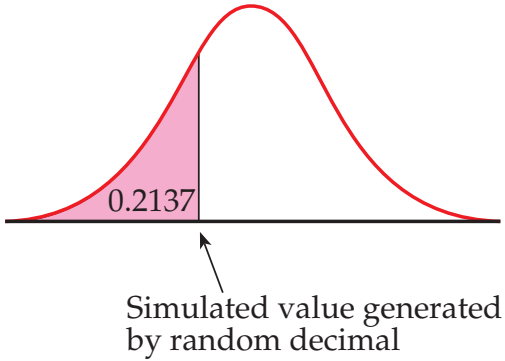
The next step is to look at the four days of the week for each trial and decide whether the days are all different. You can either do this by looking at the days, or use a formula in column J that gives 1 if the numbers in columns B to E are all different. This formula is = IF(B2=C2,0,IF(B2=D2,0, IF(B2=E2,0,IF(C2=D2,0,IF(C2=E2,0,IF(D2=E2,0,1)))))). This formula works by testing each pair in turn, and giving 0 if the pair is the same, otherwise eventually giving 1 – indicating all four days are different. At the bottom of column J,

To improve the simulation, many more than 20 trials are needed. You can download the spreadsheet 'Day of the week simulation (1000 trials).xls' from the *Theta Mathematics* student CD or from the www.mathematics.co.nz website and run it repeatedly to get a more accurate estimate. Pressing the F9 key repeats the simulation.

The theoretical probability that the members of a four-person family each have a different day of birth is $\frac{120}{} = 0.35$ (2 dp).

27.02 Simulating values from a normal distribution

Considering a normal distribution with a given mean and standard deviation, what are some typical values? The properties of the normal distribution tell us that just about all values are within 3 standard deviations of the mean, but how do we obtain values that actually reflect normally distributed data?



We can apply simulation techniques to provide as many typical values from a normal distribution as required. We generate random decimals, and change each one into an x-value for the normal distribution with given mean and standard deviation. Each random decimal is taken to be the probability from the extreme lower value of the distribution. For example, the random decimal 0.2137 would yield a value below the mean, as shown in the diagram.

We obtain the actual value by doing an 'inverse normal' conversion, assuming we know the mean and standard deviation.

The values can be obtained by using a random decimal generator in a spreadsheet and converting each one using the formula =NORMINV(probability, mean,standard deviation) where the probability is a random decimal.

Example

Simulate a sample of five values from a population that is normally distributed, with a mean of 29.58 and a standard deviation of 6.227.

Answer

Generate five random decimals and use the NORMINV formula to convert each one, with 29.58 as the mean and 6.227 as the standard deviation. In the spreadsheet, place the random decimals in column A, and the associated simulated values in column B.

Formulae

	A	B
1	**Random decimal**	**Simulated value**
2		
3	=RAND()	=NORMINV(A3,29.58,6.227)
4	=RAND()	=NORMINV(A4,29.58,6.227)
5	=RAND()	=NORMINV(A5,29.58,6.227)
6	=RAND()	=NORMINV(A6,29.58,6.227)
7	=RAND()	=NORMINV(A7,29.58,6.227)

Actual results

	A	B	C	D	E	F	G
1	**Random decimal**	**Simulated value**					
2							
3	0.441536653	28.66					
4	0.939341154	39.23		To generate another sample, press F9			
5	0.480797575	29.28					
6	0.12431125	22.40					
7	0.072113859	20.49					

The five simulated values are the numbers in column B.

28.01 Finding intersections by drawing lines

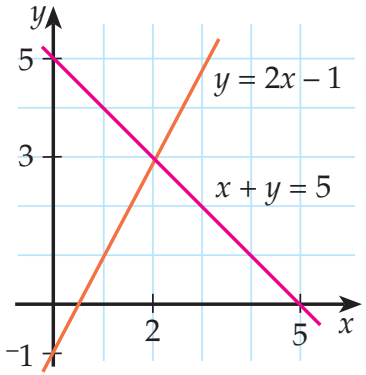
Example

Draw a graph to solve these simultaneous equations.

(1) $x + y = 5$

(2) $y = 2x - 1$

Answer



The graphs intersect at the point (2, 3).
Hence the solution is $x = 2$ and $y = 3$.

28.02 Elimination method

This method works by ‘eliminating’, or removing, one of the variables to get a single equation in one variable. The is done by either adding or subtracting the two equations. The single equation is then solved, and the result is substituted into one of the original equations to find the other variable.

Example

Solve this pair of simultaneous equations.

(1) $2x - y = 7$

(2) $3x + y = 13$

(*This shows very simple elimination by adding.*)

When writing working for solving simultaneous equations, it helps to keep the *x*s, *y*s and equals signs lined up!

Answer

Add the two equations to eliminate y. Note that the xs, ys and numbers after the equals sign are all *added*.

$$\begin{aligned} (1) \qquad & 2x - y = 7 \\ (2) \qquad & \underline{3x + y = 13} \\ (1) + (2) \qquad & 5x = 20 \\ & x = 4 \end{aligned}$$

Substitute $x = 4$ into equation (2) to find y:

$$\begin{aligned} 3 \times 4 + y &= 13 \\ 12 + y &= 13 \\ y &= 13 - 12 \\ y &= 1 \end{aligned}$$

The solution is $x = 4$, $y = 1$.

Example

Solve this pair of simultaneous equations.

(1) $3x - 5y = 23$

(2) $3x + 2y = 16$

(*This shows very simple elimination by subtracting.*)

Answer

This time *subtract* the two equations to eliminate x. Note that adding the two equations is not a good idea – we would just get $6x - 3y = 39$, which does not eliminate either x or y.

$$\begin{aligned} (1) \qquad & 3x - 5y = 23 \\ (2) \qquad & \underline{3x + 2y = 16} \\ (1) - (2) \qquad & {}^{-}7y = \ \ 7 \\ & y = {}^{-}1 \end{aligned}$$

Substitute $y = {}^{-}1$ into equation (1) to find x:

$$\begin{aligned} 3x - 5 \times {}^{-}1 &= 23 \\ 3x + 5 &= 23 \\ 3x &= 23 - 5 \\ 3x &= 18 \\ x &= 6 \end{aligned}$$

The solution is $x = 6$, $y = {}^{-}1$.

Sometimes, the **coefficients** (numbers in front of x and y) have to be adjusted before we add or subtract the two equations. We do this by first multiplying one or both equations by a suitable number, to make either the x or y terms the same.

Example

Solve this pair of simultaneous equations.

(1) $12x - y = 21$

(2) $3x + 7y = 27$

(*This requires one equation to be multiplied by a suitable value first.*)

Answer

Multiply equation (1) by 7 first, and then add the two equations to eliminate y and find x. (Note: we could just as easily multiply equation (2) by 4 first, and then subtract the two equations to eliminate x and find y.)

$$\begin{array}{ll} (1) & 12x - y = 21 \\ (2) & 3x + 7y = 27 \\ (1) \times 7 & 84x - 7y = 147 \\ (2) & 3x + 7y = 27 \\ \hline & 87x = 174 \\ & x = 2 \end{array}$$

Substitute $x = 2$ into equation (2) to find y:

$$\begin{aligned} 3 \times 2 + 7y &= 27 \\ 6 + 7y &= 27 \\ 7y &= 27 - 6 \\ 7y &= 21 \\ y &= 3 \end{aligned}$$

The solution is $x = 2, y = 3$.

28.03 Substitution method

This is the preferred method when one equation has one of the variables as its subject. This variable is substituted into the other equation.

Example

Solve this pair of simultaneous equations.

(1) $y = 2x - 3$

(2) $4x - 3y = 9$

Answer

Replace y in equation (2) with $(2x - 3)$:

$$\begin{aligned} 4x - 3y &= 9 \\ 4x - 3(2x - 3) &= 9 \\ 4x - 6x + 9 &= 9 \\ -2x &= 9 - 9 \\ -2x &= 0 \\ x &= 0 \end{aligned}$$

Substitute $x = 0$ into equation (1) to find y:

$y = 2 \times 0 - 3 = 0 - 3 = -3$

The solution is $x = 0$, $y = -3$.

29.01 Intersection of a line and a parabola

KEY POINTS

There are three possibilities for the intersection of a line and a parabola.

We look at what happens both:

- from a graphical point of view, and
- by using algebra to solve equations.

When solving an equation of a line and an equation of a parabola simultaneously, a quadratic equation always results. If the resulting quadratic equation is $ax^2 + bx + c = 0$, then:

1 $b^2 - 4ac < 0$

The parabola and line do not intersect.

No intersection.

2 $b^2 - 4ac = 0$

The line touches the parabola at one point only.

The line is a **tangent** to the parabola.

3 $b^2 - 4ac > 0$

The line cuts the parabola at two different points.

Two points of intersection.

Example

Find the points of intersection of the line $y = 2x + 1$ and the parabola with equation $y = x^2 + 1$.

Draw the line and parabola on a graph, labelling the points of intersection.

Answer

Solve simultaneously the equations

$y = 2x + 1$ (1)

$y = x^2 + 1$ (2)

Do this by substituting $x^2 + 1$ for y in the equation $y = 2x + 1$:

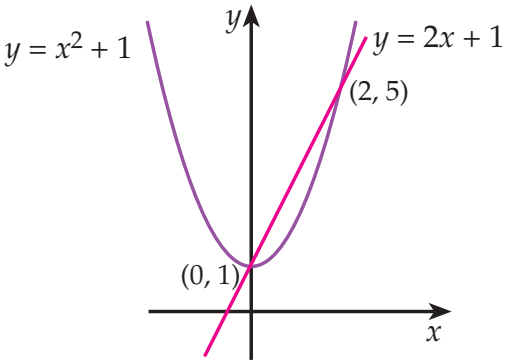
$$y = 2x + 1$$
$$x^2 + 1 = 2x + 1$$
$$x^2 + 1 - 2x - 1 = 0$$
$$x^2 - 2x = 0$$
$$x(x - 2) = 0$$
$$x = 2 \text{ or } 0$$

Substitute each of these x-values into either equation (1) or (2) to find the corresponding y-values:

$y = 2^2 + 1 = 5$

$y = 0^2 + 1 = 1$

The two points of intersection are (2, 5) and (0, 1).



Example

Use algebra to find the points of intersection of the line $3x - y + 11 = 0$ and the parabola $y = x^2 + 4x + 5$.

Answer

Solve simultaneously the equations

$3x - y + 11 = 0 \qquad (1)$

$y = x^2 + 4x + 5 \qquad (2)$

Do this by substituting $x^2 + 4x + 5$ for y in equation (1).

Equation (1) becomes:

$$3x - (x^2 + 4x + 5) + 11 = 0$$
$$3x - x^2 - 4x - 5 + 11 = 0$$
$$^{-}x^2 - x + 6 = 0$$
$$x^2 + x - 6 = 0$$
$$(x - 2)(x + 3) = 0$$
$$x = 2 \text{ or } ^{-}3$$

Substitute each of these x-values into either equation (1) or (2) to find the corresponding y-values:

$y = 2^2 + 4 \times 2 + 5 = 17$

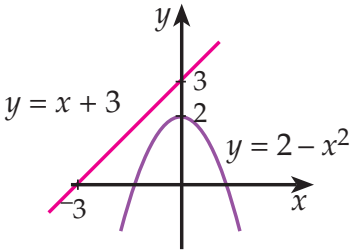
$y = (^-3)^2 + 4 \times (^-3) + 5 = 2$

The two points of intersection are $(2, 17)$ and $(^-3, 2)$.

Example

Draw the graphs of $y = 2 - x^2$ and $y = x + 3$ on the same diagram. Use algebra to show that the curve and line do not intersect.

Answer



Attempt to solve simultaneously:

$$x + 3 = 2 - x^2$$

$$x^2 + x + 1 = 0$$

This quadratic equation has discriminant $b^2 - 4ac = 1^2 - 4 \times 1 \times 1 = {}^{-}3$, and hence has no solution.

29.02 Intersection of a line and a circle

KEY POINTS

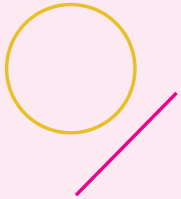
There are three possibilities for the intersection of a line and a circle (as is the case for a line and a parabola).

When solving an equation of a line and an equation of a circle simultaneously, a quadratic equation always results. If the resulting quadratic equation is $ax^2 + bx + c = 0$, then:

1 $b^2 - 4ac < 0$

The circle and line do not intersect.

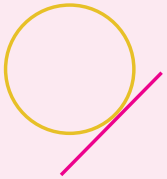
No intersection.



2 $b^2 - 4ac = 0$

The line touches the circle at one point only.

The line is a **tangent** to the circle.



3 $b^2 - 4ac > 0$

The line cuts the circle at two different points.

Two points of intersection.

Example

Show that the line $y = 4$ is a tangent to the circle $(x - 3)^2 + y^2 = 16$. Find the co-ordinates of the point of contact.

Answer

Solve simultaneously by substituting $y = 4$ into the circle equation:

$$(x - 3)^2 + (4)^2 = 16$$
$$x^2 - 6x + 9 + 16 = 16$$
$$x^2 - 6x + 9 = 0$$

This quadratic equation has discriminant $(^{-}6)^2 - 4 \times 1 \times 9 = 0$, and hence has one repeated real root. Thus, the line touches the circle, and is therefore a tangent.

$$x^2 - 6x + 9 = 0$$
$$(x - 3)^2 = 0$$
$$x = 3 \text{ and } y = 4$$

The point of contact is (3, 4).

Example

Find the co-ordinates of the point(s) of intersection of the line $y = x - 3$ and the circle $x^2 + y^2 = 5$. Illustrate by drawing a diagram.

Answer

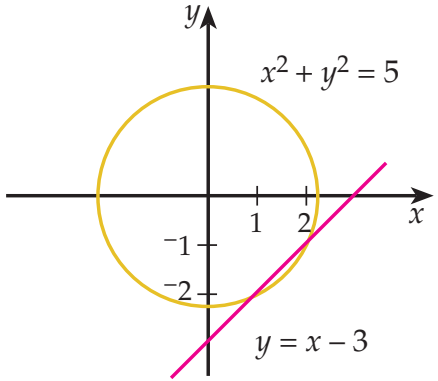
Solve simultaneously by substituting $y = x - 3$ into the circle equation:

$$\begin{aligned} x^2 + y^2 &= 5 \\ x^2 + (x-3)^2 &= 5 \\ x^2 + x^2 - 6x + 9 &= 5 \\ 2x^2 - 6x + 4 &= 0 \\ x^2 - 3x + 2 &= 0 \\ (x-2)(x-1) &= 0 \\ x = 2 \text{ or } x &= 1 \end{aligned}$$

There are two points of intersection. To find the y-co-ordinates, substitute each of $x = 2$ and $x = 1$ in turn into the line equation $y = x - 3$:

$y = 2 - 3 = {}^{-}1$ or $y = 1 - 3 = {}^{-}2$

The two points of intersection are $(2, {}^{-}1)$ and $(1, {}^{-}2)$.



29.03 Intersection of a line and a hyperbola

KEY POINTS

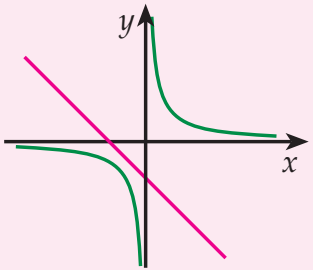
There are three possibilities for the intersection of a line and a hyperbola.

When solving an equation of a line and an equation of a hyperbola simultaneously, a quadratic equation always results. If the resulting quadratic equation is $ax^2 + bx + c = 0$, then:

1 $b^2 - 4ac < 0$

The hyperbola and line do not intersect.

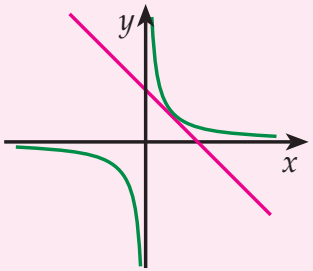
No intersection.



2 $b^2 - 4ac = 0$

The line touches the hyperbola at one point only.

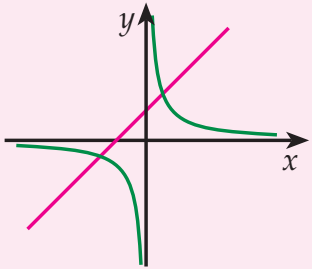
The line is a **tangent** to the hyperbola.



3 $b^2 - 4ac > 0$

The line cuts the hyperbola at two different points.

Two points of intersection.



Example

Find the co-ordinates of the point(s) of intersection of the line $y = 2x - 1$ and the hyperbola $xy = 15$. Illustrate by drawing a diagram.

Answer

Solve simultaneously by substituting $y = 2x - 1$ into the hyperbola equation:

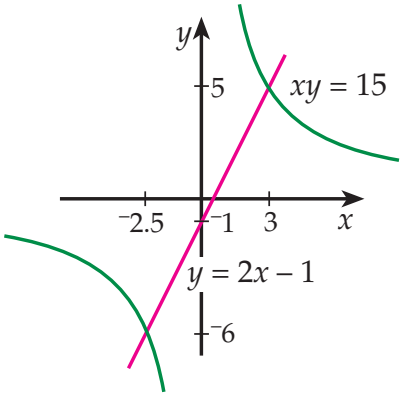
$$\begin{aligned} xy &= 15 \\ x(2x-1) &= 15 \\ 2x^2 - x - 15 &= 0 \\ (2x+5)(x-3) &= 0 \end{aligned}$$

$$x = \frac{-5}{2} \text{ or } x = 3$$

There are two points of intersection. To find the y-co-ordinates, substitute each of $x = \frac{-5}{2}$ and $x = 3$ in turn into the line equation $y = 2x - 1$:

$$y = 2 \times \frac{-5}{2} - 1 = -6 \quad \text{or} \quad y = 2 \times 3 - 1 = 5$$

The two points of intersection are $\left(\frac{-5}{2}, -6\right)$ and $(3, 5)$.



Example

Show that the line $y = 4x + 8$ is a tangent to the hyperbola $y = \frac{^-4}{x}$. Find the co-ordinates of the point of contact.

Answer

Solve simultaneously:

$$4x + 8 = \frac{^-4}{x}$$

$$4x^2 + 8x = {}^-4$$

$$4x^2 + 8x + 4 = 0$$

$$x^2 + 2x + 1 = 0$$

$$(x + 1)^2 = 0$$

This quadratic equation has one (repeated) solution only. Thus, the line touches the hyperbola, and is therefore a tangent.

The x-co-ordinate of the point of contact is $^-1$. To find the y-co-ordinate, substitute $x = {}^-1$ into either the line or hyperbola equation:

$$y = \frac{^-4}{^-1} = 4$$

The point of contact is $(^-1, 4)$.

29.04 Applications of non-linear simultaneous equations

Example

The water content of a soil sample is defined as the fraction:

$$\frac{\text{volume of water in the sample}}{\text{total volume of water, soil and air in the sample}}.$$

Two to three days after rain or irrigation, the water content in the soil is typically 0.35 and, after 100 days without adding any more water, the water content is effectively 0.

Two possible models for water content are (1) a rational function (has a hyperbola as its graph), and (2) a linear function. Consider the following.

(1) A rational function: $w = \frac{1.225}{t+1}$; where w is the water content and t is the time elapsed in days since the sample was last watered.

(2) A linear function that satisfies the pairs ($t = 2.5$, $w = 0.35$) and ($t = 100$, $w = 0$).

a Write the equation of the linear function.

b Explain why the rational function is not appropriate to use when the soil sample has been first watered.

c Determine the second time when the two models give the same water content.

Answer

a Use co-ordinate geometry to determine the equation of the line that passes through (2.5, 0.35) and (100, 0).

Note that the rule $y - y_1 = m(x - x_1)$ gives the equation of a line with gradient m that passes through the point (x_1, y_1).

$$w - 0 = \frac{0 - 0.35}{100 - 2.5}(t - 100)$$

$$w = \frac{-0.35}{97.5}(t - 100)$$

$$w = \frac{-7}{1950}(t - 100)$$

This equation can be rearranged to $1950w + 7t = 700$.

b When the soil sample is first watered, $t = 0$. Substituting $t = 0$ into the rational function gives $w = 1.225$, or 122.5%. It is not possible for the percentage of water in any kind of substance to exceed 100%.

c Solve the pair of simultaneous equations:

(1) $w = \dfrac{1.225}{t+1}$ and (2) $1950w + 7t = 700$.

Two approaches are possible.

The first approach is to solve manually:

$$\frac{1.225}{t+1} = \frac{-7}{1950}(t - 100)$$

$$-341.25 = (t+1)(t-100)$$

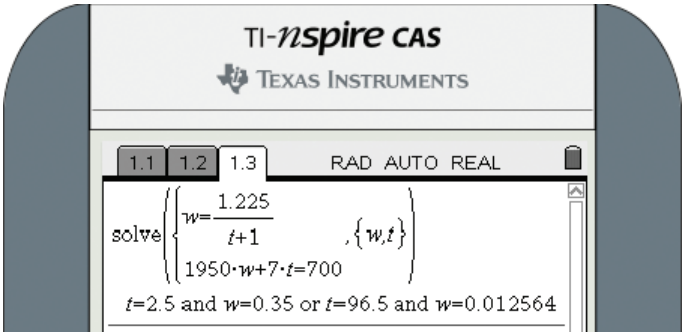
$$-341.25 = t^2 - 99t - 100$$

$$t^2 - 99t + 241.25 = 0$$

Then use the quadratic formula, or factorise:

$(t - 2.5)(t - 96.5) = 0$

$t = 2.5$ or 96.5

We want the second of these two times, so the answer is at 96.5 days.

The second approach is to solve using technology (CAS, or equation-solving software):